The art of Counting
Combinatorics

What is counting?
1. say numbers: to say numbers in order, usually starting at one
2. add up: to add things up to see how many there are or to find the value of an amount of money
3. include: to include somebody or something in a calculation
4. consider or be considered: to consider somebody or something, or be considered, in a particular way or as a particular thing
5. be of importance: to be of importance or value
6. have a value: to have a specific value
7. music dance keep time: to keep time by counting beats

• [14th century. The noun is via old French conte; the verb directly from Old French conter “to reckon (estimaciones),” from Latin computare, literally “to reckon together.”] Microsoft® Encarta® Reference Library 2003. © 1993-2002 Microsoft Corporation. All rights reserved.

• Ethymologies
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– com (con) y putare (pensar)

– One famous quote by Terence reads: “Homo sum, humani nihil a me alienum puto”, or “I am human, nothing that is human is alien to me.” This appeared in his play Heauton Timorumenos.

What is counting?

• Synonims
– calculation, number crunching, reckoning, computation

– total, sum total, sum, amount, tally

– Tally:
– Paleolithic tally bones with numerical notches.
– Mesopotamia:

What is counting?

• An encyclopedic account of how counting was developed:

HISTORIA UNIVERSAL DE LAS CIFRAS
de IFRAH, GEORGES
ESPASA-CALPE, S.A.
ISBN: 8423997308

G. Ifrah: The Universal History of Numbers.
Strategies for counting

• Intension vs. Extension
  – Extension: \[\begin{align*}
1 &\mapsto \text{Elem}1 \\
2 &\mapsto \text{Elem}2 \\
N &\mapsto \text{Elem}N
\end{align*}\]
  – Intension: \[\begin{align*}
\text{Elem}1 &\mapsto \text{Description}1 \\
\text{Elem}2 &\mapsto \text{Description}2 \\
\text{Elem}N &\mapsto \text{Description}N
\end{align*}\]

Bijection

• One to one correspondence between sets of equal size.
  – Hilbert’s hotel
  
Bijection

• Compression of files.
  – No Lossless compression of files.

Bijection

• Counting with the pigeon hole + description.
  – There must be at least two people in London with the same number of hairs on their head.
  – A typical head has around 150,000 hairs
  – Birthday problem with N>365
  – Collisions in a Hash table.
  – Number of keys exceed the number of indices in the array.

Bijection

• One to one correspondence between sets.
  • Examples:
    – Multiples of 7 between 39 and 292
      \[i \leftrightarrow 7(5+i)\]
      \[\begin{align*}
1 &\leftrightarrow 42 \\
2 &\leftrightarrow 49 \\
\vdots &\rightarrow 36 \leftrightarrow 287
\end{align*}\]
  
Examples taken from VELÉZ, HERNÁNDEZ, Cálculo de Probabilidades
Bijection

• Example Extension/Intension:
  - Compute the number of games on a tennis championship with 84 players
    - First round -> 42 games (42 players), Second -> 21g (21p), third -> 10g (10p), fourth -> 5g (5p), fifth -> 3g (3p), sixth -> 1g (1p)
    - Total: 42 + 21 + 10 + 5 + 3 + 1 + 1 = 83
  - Algorithm: Play matches until one player remains
  - Notice that for $n$ players -> $n-1$ games.

Examples taken from VÉLEZ, HERNÁNDEZ, Cálculo de Probabilidades

Bijection

• Numerable sets -> Cardinality of N
  - We admit infinite sets as long as are numerable.
  - R are not admitted:
    - Cantors diagonalization:
      $n_1 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...)$
      $n_2 = (0, 1, 4, 5, 6, 7, 8, 9, 10, 11, ...)$
      $n_3 = (0, 5, 6, 7, 8, 9, 10, 11, 12, 13, ...)$
      $n_4 = (0, 1, 2, 5, 6, 7, 8, 9, 10, 11, ...)$
      $n_5 = (0, 1, 4, 5, 6, 7, 8, 9, 10, 11, ...)$
      $n_6 = (0, 3, 6, 7, 8, 9, 10, 11, 12, 13, ...)$
      $n_7 = (0, 1, 7, 8, 9, 10, 11, 12, 13, 14, ...)$
      $n_8 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...)$
      $n_9 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...)$
      $n_{10} = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...)$

Probability of an infinite set?

Product rule

• Given the sets $A_1, \ldots, A_k$ each with $n_1, \ldots, n_k$ elements, the cartesian product $A_1 \times A_2 \times \cdots \times A_k$ has $n_1 n_2 \cdots n_k$ elements

Examples:
  - Matrix of linear algebra. Number of coefficients when you have $m$ equations and $n$ unknowns
    $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$
Product rule

• Examples:
  – Bigrams or n-grams

\[ 24 \cdot 24 = 576 \]
\[ 24 \cdots 24 = 24^n \]

Product rule

• Possible Applications of n-grams:
  – Prob of words of two letters with one vocal

\[ \Pr = \frac{\text{Count of ways for result}}{\text{Count of all possible results}} \]

\[ \Pr = \frac{\text{Number of times result appears}}{\text{Count of the number of trials}} \]

Product rule

• Example:
  – How many different words of 5 different letters can be formed with the letters A,E,I,L,M,N,P with a vocal in first position

\[
\begin{array}{ccc}
\text{A} & \text{E} & \text{I} \\
\text{X} & \text{X} & \text{X} \\
3 & 6 & 5
\end{array}
\]

\[ 3 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 1080 \]

Usual Patterns

• Counting permutations:
  – How to order \( n \) different objects?

\[
\{a_1, a_2, \ldots, a_n\} \quad n(n-1)(n-2)\cdots 1 = n! \\
\{a_1, a_2, \ldots, a_n\} \setminus \{a_i\} \quad \text{Down} \\
\downarrow \\
\downarrow \\
X_X_X \cdots X \\
\frac{n!}{(n-1)!} \quad 2 \quad 1
\]

Usual Patterns

• Orderings/Permutations
  – Application of the product rule
  – **Distinguishable** objects may be arranged in various different orders

\[
\text{Number of possible arrangements} = (n-1)(n-2)\cdots 1 \\
\text{Definition of Permutation} \\
\text{on } n \text{ elements uses a } \begin{pmatrix} n \end{pmatrix} \text{ matrix, where the first row is } (123\ldots n) \text{ and the second row is the new arrangement.} \\
\text{For example, the permutation which switches elements 1 and 2 and fixes 3 would be written as} \\
\begin{pmatrix} 1 & 2 & 3 \\
1 & 2 & 3
\end{pmatrix}
\]

Usual Patterns

• Counting permutations/Comments:
  – How to order \( n \) different objects in \( r \) positions?

\[
\{a_1, a_2, \ldots, a_n\} \quad \text{Down} \\
\downarrow \\
\downarrow \\
X_X_X \cdots X \\
\frac{n!}{(n-r)!} \quad \frac{n!}{(n-r)!} \quad 2 \quad 1
\]

Is there a relation with the indistinguishable case?
Usual Patterns

• Counting permutations:
  – Example: Possible noncoincident birthdays of \( r \) persons.

\[
\frac{365!}{365-(r-2)\cdot(365-(r-1))}\cdot\frac{363!}{365-(r-1)\cdot(365-(r-2))}\cdot\ldots\cdot\frac{365-(r-2)!}{365}\cdot\frac{364!}{365-(r-1)!}\cdot\ldots\cdot\frac{1!}{365}
\]

– Note: This example will be used for the coincidence problem, and for deriving the Poisson pdf.

Usual Patterns

• Counting permutations without repetition:
  – Another way of setting the problem.

\[
\text{Possible configurations } = \frac{365!}{(365-r)!}
\]

Usual Patterns

• Counting permutations with repetition:
  – How to order \( n \) different objects with repetition?

\[
\left\{ a_1, a_2, \ldots, a_r \right\}
\]

\[
\left\{ a_1, a_2, \ldots, a_r \right\}
\]

\[
\downarrow
\]

\[
X \ X \ X \ \ldots \ X \ X
\]

\[
\sum_{r=0}^{n} \binom{n}{r} r^n
\]

Usual Patterns

• Internet’s Diameter:

\[
\bar{d} = \frac{\sum_{i=1}^{N} k_i \cdot \min \{ k_i \}}{\sum_{i=1}^{N} k_i}
\]

\[
\frac{\bar{d}}{\log N} = 19
\]

\[
\frac{d}{\log \frac{N}{\log \bar{d}}} = 19
\]

But \( 10^{19} \gg 10^9 \), what does it mean?

Maximum path from one node to the other.
Usual Patterns

- What does \( d = \frac{\log N}{\log 10} \approx 19 \) mean?

- Number of atoms in the universe is \( N = 10^{78} \)

- The logarithm gives \( d \approx \frac{\log_{10} N}{\log_{10} 10} \approx 32 \)

  • Notice: the logarithm gives the number of decimal places needed to write the number.
  - It is a description of the number

\[ \text{Mean number of links/node} = \frac{\text{Number of nodes}}{N} \]

Usual Patterns

- Six Degrees
  - **Six degrees of separation** is the hypothesis that anyone on Earth can be connected to any other person on the planet through a chain of acquaintances with no more than five intermediaries.

  ![Six Degrees Diagram](image1.png)

Six Degrees/Example

- How do you get from Muhammad Ali to Bob Dylan?
  - 1: Muhammad Ali’s manager for most of his career was Angelo Dundee.
  - 2: Angelo Dundee also managed Heavyweight Champion Jimmy Ellis.
  - 3: In 1964, Jimmy Ellis was defeated by Rubin “Hurricane” Carter.
  - 4: The song *Hurricane* was written about Rubin Carter and appears on Bob Dylan’s album *Desire*.

![Small-world Network](image2.png)

Usual Patterns

- Comparison:
  - Number of orderings \( n \) different objects selected \( r \) times with and without repetition?

\[ n^r \text{ vs.} \frac{n!}{(n-r)!} \]

Usual Patterns

- Counting permutations/indistinguishability:
  - **Some** indistinguishable objects.
  - \( n \) objects each with \( \{a_1, a_2, \ldots, a_n\} \) elements
  - Notation: \( \{a_j\} \text{Object k of class j} \)
  - Notice that for a given permutation:
    \[ \{x, a_1, a_2, \ldots, x, a_1, a_2, \ldots\} \rightarrow \{x, a_1, a_2, \ldots, x, a_1, a_2, \ldots\} \]

  • The \( \{n\} \) permutations are equivalent, therefore have to be discounted

\[ \text{Different Permutations} \rightarrow \frac{n!}{m_1!} \]
Usual Patterns

• Counting permutations:
  – General Formula
  \[ \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!} \quad n = n_1 + n_2 + \ldots + n_k \]
  – Different ways to order a collection of \( n \) objects with \( k \) subsets of indistinguishable objects.

• Example: With the symbols \{a,a,a,b,b,c,c,d\} how many words can be constructed?
  – Note that the same word can be constructed in different ways.
  – The number of words is:
  \[ \frac{8!}{2! \cdot 2! \cdot 3!} = 1680 \]

Examples taken from VÉLEZ, HERNÁNDEZ, Cálculo de Probabilidades

Usual Patterns

• Models of the equation
  – Thermodynamics: \( \{x\} \) particles in a given state. How many different realizations can we have?
  – Language Models
  – Derivation of the Entropy (Thermodynamics/Information Theory)

\[ (n_1, n_2, \ldots, n_k) = \frac{[n_1 + n_2 + \ldots + n_k]}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!} \]

Usual Patterns

• multinomial example (Rao, 1973)

There are \( n=197 \) animals classified into one of 4 categories: \((y_1, y_2, y_3, y_4) = (125, 18, 20, 34)\)

Number of different orderings is

\[ \frac{n!}{y_1! \cdot y_2! \cdot y_3!} \]

Objective of the paper: test if the observations relate to the probability model.

Usual Patterns

• Language Models

Does a text correspond to an author?

Apriori probability of using a word: \( \{p_1, p_2, \ldots, p_n\} \)

A text has the word frequencies: \( \{n_1, n_2, \ldots, n_n\} \)

Number of different books with the same word frequencies

\[ \frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_n!} \]

Note: Does not take into account the probability of a word.

Usual Patterns

- **Language Models**
  Does a text correspond to an author?
  - Apriori probability of using a word: \( P_1, P_2, \ldots, P_n \)
  - A text has the word frequencies: \( n_1, n_2, \ldots, n_k \)
  - Probability that the text was written by the author

Meanings of model

- **Engineering vs. Mathematics/logic**

Usual Patterns

- **Subsets**
  - Given a set of \( n \) elements, how many subsets of cardinality \( r \) can be made?
  - Examples: How many possibilities when:
    - Throw a die \( n \) times and get \( r \) faces
    - Sell \( n \) sits in plane and have \( r \) cancelations
      - Overbooking, D’alembert’s mistake.

Usual Patterns

- **Subsets**
  - How many possibilities when: Sell \( n \) sits in plane and have \( r \) cancelations
    - Notice 2 points:
      - Temporal order of booking is not important
      - Sets are ‘bookings’, not ‘cancelations’
      - D’alembert’s mistake.

Usual Patterns

- \( n \) sits in plane and have \( r \) cancelations
  - Define a set of indices:
  - Possible sets of bookings:
    - Why \((n-r)\)? (orders?)
    - If each booking is different
    - But in a set order is not important
  - Is there a difference between the sets of cancelations?
Usual Patterns
- \( n \) sits in plane and have \( r \) cancelations
\[ \frac{n!}{r!(n-r)!} \]

Discounting the orders of cancelations
\[ \text{Total} = \frac{n!}{r!(n-r)!} \]

Usual Patterns
- Subsets: General expression
- How many possibilities when: Sell \( n \) sits in plane and have \( r \) cancelations
\[ C(n, k) = \binom{n}{k} \frac{n!}{(n-k)!k!} \]
- Read \( n \ choose \ k \)

Usual Patterns
- Properties of \( C(n, k) \):
  - Binomial Theorem
  \[ (x+b)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} b^k \]

Usual Patterns
- Properties of \( C(n, k) \):
  - Pascal’s triangle

Usual Patterns
- Properties of \( C(n, k) \):
  - Recursive Computation
Usual Patterns

• Properties of C(n,k):
  – Binomial Theorem/Important consequence
  – Total number of subsets of a set.

\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = \binom{3}{0} a^3 + \binom{3}{1} a^2b + \binom{3}{2} ab^2 + \binom{3}{3} b^3\]

\[
\text{Number Subsets} = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}
\]

Usual Patterns

• Properties of C(n,k):
  – Total number of subsets of a set.
  • General Expression

\[N = (1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} = 2^n\]

Usual Patterns

• Properties of C(n,k):
  – Total number of subsets of a set.
  • Another derivation

\[N = 2^{22} \cdots \frac{2}{n} = 2^n\]

Usual Patterns

• Simple application:
  – Number of links in a graph

\[\binom{n}{2} - n(n-1)/2\]

Stirling’s Formula

• Justification of Stirling’s Formula
  – Helps to deal and compute enormous numbers

\[n! = \sqrt{2\pi n} n^{n\frac{1}{2}} e^{-n} \frac{n!}{(n-i)!}\]

Stirling Formula

• Short derivation

\[\ln n! = \ln(1) + \ln(2) + \cdots + \ln(n)\]

\[\ln n! = n\ln n - n + \frac{1}{2}\ln(2\pi) + \cdots\]

\[\ln n! = \sqrt{2\pi n} n^{n\frac{1}{2}} e^{-n} n!\]

\[e^{-n} n! = \sqrt{2\pi n} n^{n\frac{1}{2}} e^{-n}\]
Stirling Formula

• Short derivation

\[ n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \]

Important:

- **Ratio** not difference!

Exercises

• Ten persons select a paper each from an urn with ten papers. One paper has the prize. How many permutations are possible if the winner has made selection 'i'.

Exercises

• Secretary problem (dowry, perfect husband, etc.)
- You have 20 candidates and examine them sequentially.
- How many possibilities do you have of selecting the best, if your strategy is to examine the first 10, and then select the best of the last 10.