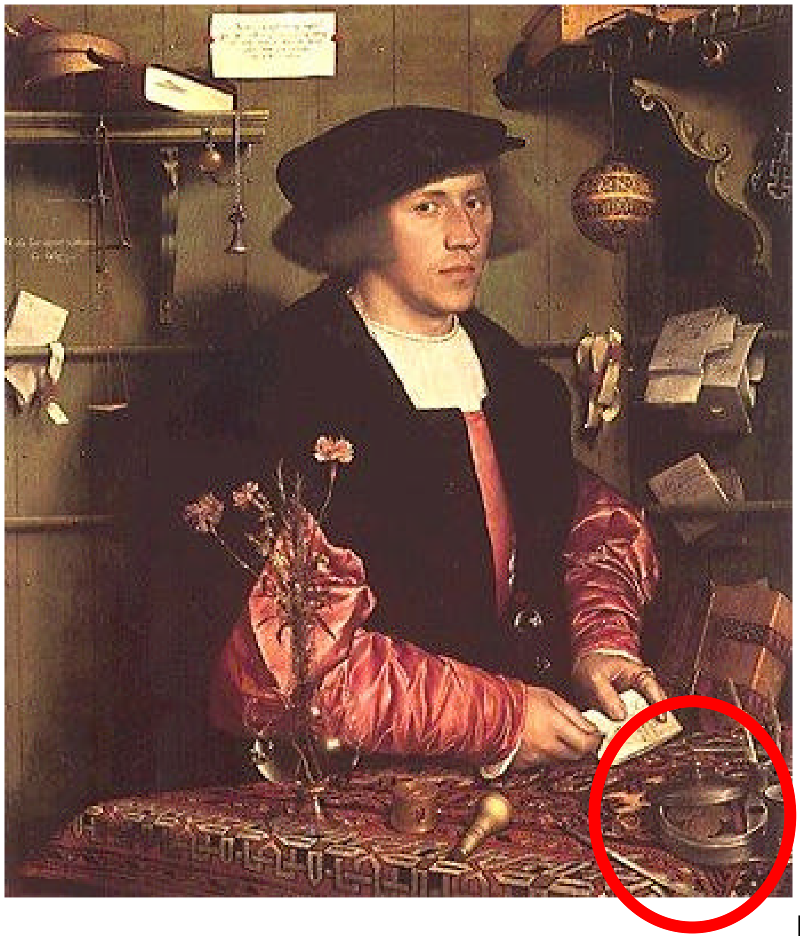
The art of Counting
Combinatorics



Portrait of Georg Gisz

What is counting?

1. **say numbers**: to say numbers in order, usually starting at one
2. **add up**: to add things up to see how many there are or to find the value of an amount of money
3. **include**: to include somebody or something in a calculation
4. consider or be considered: to consider somebody or something, or be considered, in a particular way or as a particular thing
5. be of importance: to be of importance or value
6. have a value: to have a specific value
7. music dance keep time: to keep time by counting beats

- [14th century. The noun is via old French *conte* ; the verb directly from Old French *conter* “*to reckon (estimaciones*),” from Latin *computare* , literally “to reckon together.”]**Microsoft® Encarta® Reference Library 2003.**
- com (con) y putare (pensar)

What is counting?

- Ethymologies
  - [14th century. The noun is via old French *conte* ; the verb directly from Old French *conter* “to reckon (estimaciones),” from Latin *computare* , literally “to reckon together.”]**Microsoft® Encarta® Reference Library 2003. © 1993-2002 Microsoft Corporation. All rights reserved.**
  - com (con) y putare (pensar)
  - One famous quote by Terence reads: "Homo sum, humani nihil a me alienum puto", or "I am human, nothing that is human is alien to me." This appeared in his play Heauton Timorumenos.

What is counting?

- Synonims
  - **calculation**, number crunching, reckoning, computation
  - **total**, sum total, sum, amount, tally
- Tally:
  - **Paleolithic tally bones with numerical notches.**

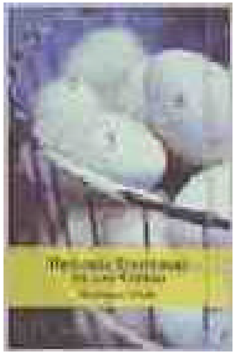
http://www.hunmagyar.org/turan/magyar/tor/carp

  - **Mesopotamia:**

http://www.edp.ust.hk/math/history/2/2_2

What is counting?

- An encyclopedic account of how counting was develloped:



HISTORIA UNIVERSAL DE LAS CIFRAS

de IFRAH, GEORGES

ESPASA-CALPE, S.A.

ISBN: 8423997308

G. Ifrah: The Universal History of Numbers.
John Wiley & Sons, New York 2000 [ISBN 0-471-39340-1]

Strategies for counting

- Intension vs. Extension
  - Extension:
    - Direct link or enumeration of each of the elements of a set
  - Intension:
    - Description of the elements of a set by means of the properties
  - Combinatorics:
    - Counting without enumeration.

Strategies for counting

- Intension vs. Extension
  - Extension:

$$\begin{cases} 1 \leftrightarrow Elem1 \\ 2 \leftrightarrow Elem2 \\ \quad\vdots \\ N \leftrightarrow ElemN \end{cases}$$

  - Intension:

$$Description \leftrightarrow \begin{cases} Elem1 \\ Elem2 \\ \quad\vdots \\ ElemN \end{cases}$$

This tapestry, created in the late 1400s, is in the collection of the Cluny Museum in Paris, France.

A painting by the American Edward Hicks (1780–1849), showing the animals boarding Noah's Ark two by two.

Bijection

- One to one correspondence between sets of equal size.

VOL. 75, NO. 1, FEBRUARY 2002 39

Proof Without Words:
The Pigeonhole Principle

RAN LIBESKIND-HADAS
Harvey Mudd College
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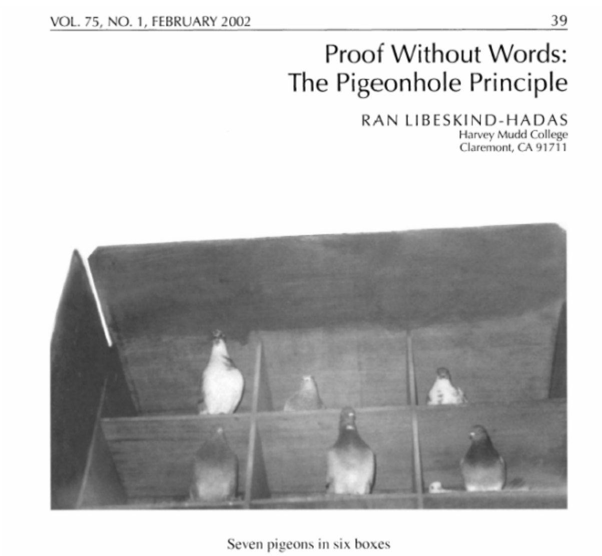
- Application to infinite sets
- Hilbert's hotel

Bijection

- Compression of files.
  - No Lossless compression of files.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Proof Without Words:
The Pigeonhole Principle

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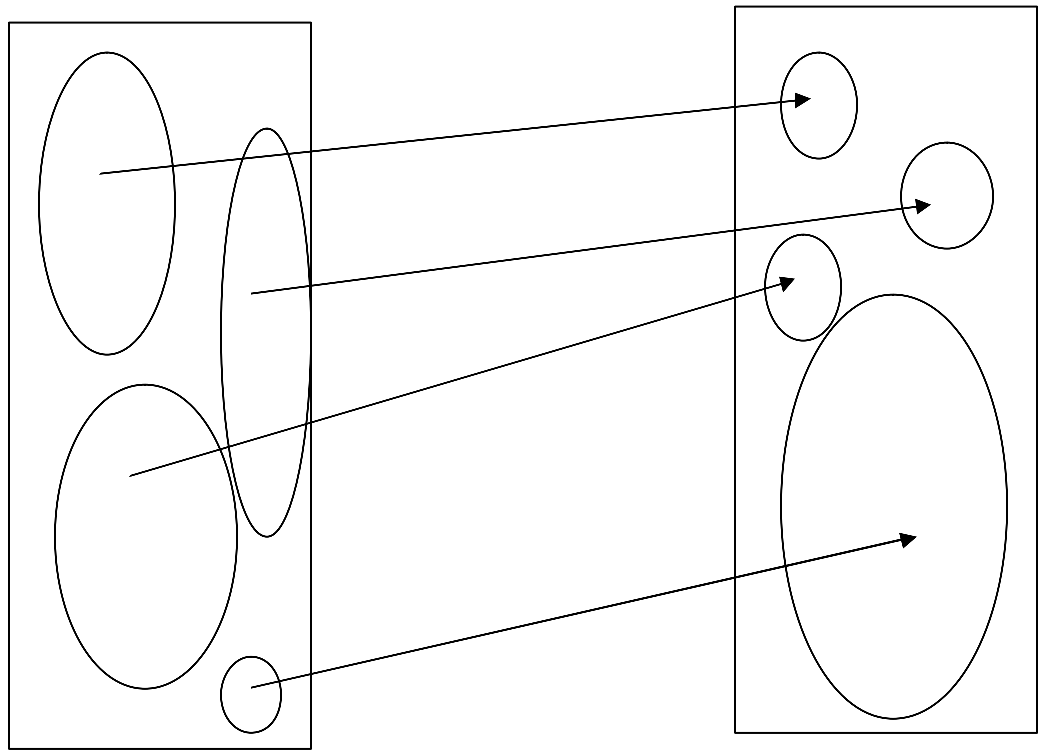
Seven pigeons in six boxes



Count all possible files of 1M and assign them a number

Bijection

- Counting with the pigeon hole+description.
  - There must be at least two people in London with the same number of hairs on their head.
    - A typical head has arround 150.000 hairs
  - Birthday problem with N>365
  - Collisions in a Hash table.
    - Number of keys exceed the number of indices in the array.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Proof Without Words:
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Bijection

- One to one correspondence between sets.
- Examples:
  - Multiples of 7 between 39 and 292

$$i \leftrightarrow 7(5+i)$$

$$\begin{gathered} 1 \leftrightarrow 42 \\ 2 \leftrightarrow 49 \\ \vdots \\ 36 \leftrightarrow 287 \end{gathered}$$

Examples taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

Bijection

- Examples:
  - Multiples of 6 with 3 cifers that beging with 4
    - Notice the structure of the problem:

$$402 = 6*67 = 6*(66+1)$$

$$498 = 6*83 = 6*(66+17)$$

$$1 \leftrightarrow 402$$

$$2 \leftrightarrow 408$$

$$\vdots$$

$$36 \leftrightarrow 498$$

$$i \leftrightarrow 6(66+i)$$

Examples taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

Bijection

- Example Extension/Intension:
  - Compute the number of games on a tennis championship with 84 players
    - First round->42 games (42players), Second->21g (21p), third->10g (11p),fourth->5g (6p),fifth->3g (3p), sixth->1g (2p), seventh->1g (1p)
    - Total: 42+21+10+5+3+1+1=83
  - Algorithm: Play matches until one player remains->Bijection between eliminated players and games.
    - Notice that for ***n*** players->***n-1*** games.

Examples taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

Bijection

- Numerable sets->Cardinality of N
  - We admit infinite sets as long as are numerable.
  - R are not admited
    - Cantors diagonalization:

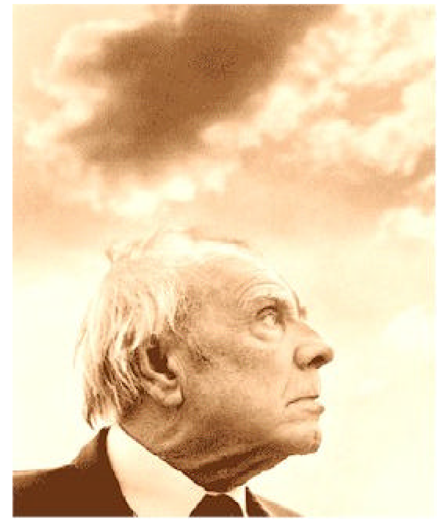
*n*1 = (0, **2**	2	3	9	5	3	5	4	5	1...)
*n*2 = (0, 1	**3**	5	6	4	6	6	8	6	6...)
*n*3 = (0, 9	6	**1**	4	4	7	2	7	8	1...)
*n*4 = (0, 5	1	6	**5**	5	3	1	9	2	8...)
*n*5 = (0, 1	1	1	4	**5**	8	1	4	1	3...)
*n*6 = (0, 3	6	6	2	5	**8**	2	3	8	9...)
*n*7 = (0, 1	1	7	7	7	9	**8**	4	9	6...)
				...					
*n*0 = (0, **3**	**4**	**5**	**6**	**6**	**9**	**9**	………………....)		

Bijection

- Numerable sets->Biblioteca de Babel

Borges' *Biblioteca de Bábel* (exc.)

l universo (que otros llaman la Biblioteca) se compone de un **número** **ndefinido**, y **tal vez infinito**, de galerías hexagonales, con vastos pozos de entilación en el medio, cercados por barandas bajísimas. Desde cualquier exágono, se ven los pisos inferiores y superiores: interminablemente. La stribución de las galerías es invariable. Veinte anaqueles, a cinco largos naqueles por lado, cubren todos los lados menos dos; su altura, que es la de los sos, excede apenas la de un bibliotecario normal. Una de las caras lires da a un ngosto zaguán, que desemboca en otra galería, idéntica a la primera y a todas. izquierda y a derecha del zaguán hay dos gabinetes minúsculos. Uno permite ormir de pie; otro, satisfacer las necesidades finales. Por ahí pasa la escalera spiral, que se abisma y se eleva hacia lo remoto. En el zaguán hay un espejo, ue fielmente duplica las apariencias. Los hombres suelen inferir de ese espejo ue la Biblioteca no es infinita (¿si lo fuera realmente, ¿a qué esa **duplicación** usoria?); yo prefiero soñar que las superficies bruñidas figuran y prometen el finito ... La luz procede de unas frutas esféricas que llevan el nombre de mparas. Hay dos en cada hexágono: transversales. La luz que emiten es suficiente, incesante.



Bijection

- Numerable sets->Biblioteca de Babel

$$book1 \rightarrow 0000000001$$

$$book2 \rightarrow 0000000002$$

$$\vdots$$

$$bookN \rightarrow 9999999999$$

$$\vdots$$

$$book\infty \rightarrow \infty\,?$$

Probability of an infinite set?

Product rule

- Given the sets $A_1, \cdots A_k$ each with $n_1, \cdots n_k$ elements, the cartesian product $A_1 \times A_2 \cdots \times A_k$ has $n_1 n_2 \cdots n_k$ elements

Feller An Introduction to Probability …..

Examples taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

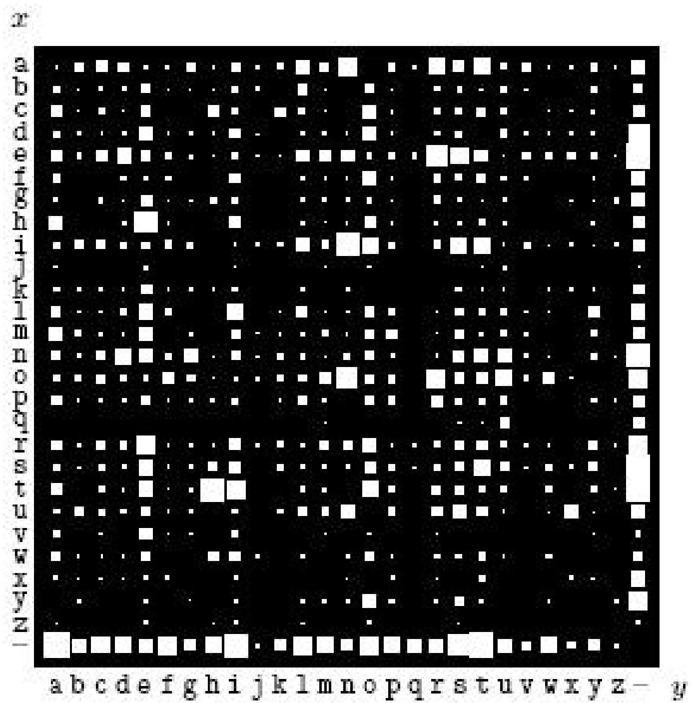
Product rule

- Examples:
  - Matrix of linear algebra. Number of coefficients when you have m equations and n unknowns

$$\begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

Product rule

- Examples:
  - Bigrams or n-grams



$$24 \cdot 24 = 576$$

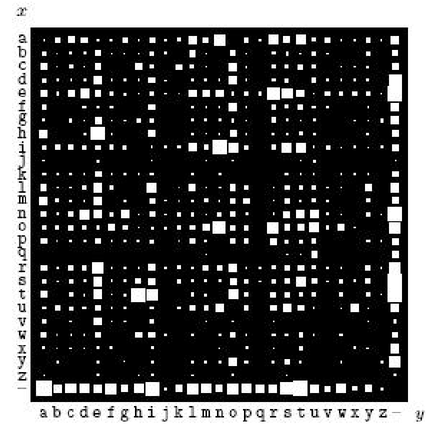
$$24 \cdot \cdots \cdot 24 = 24^n$$

http://www.inference.phy.cam.ac.uk/mackay/itila/book.html

Product rule

- Posible Aplications of n-grams:
  - Prob of words of two letters with one vocal

$$\Pr = \frac{Count\ of\ \ ways\ \ for\ a\ result}{Count\ of\ \ all\ \ possible\ results}$$



$$\Pr = \frac{Number\ of\ \ times\ a\ result\ appears}{Count\ of\ \ the\ number\ of\ \ trials}$$

Product rule

- Example:
  - How many different words of 5 different letters can be formed with the letters A,E,I,L,M,N,P with a vocal in first position

$$\left(A, E, I\right)$$

$$\downarrow$$

$$\underbrace{X}_{3}\,\underbrace{X}_{6}\,\underbrace{X}_{5}\,\underbrace{X}_{4}\,\underbrace{X}_{3}$$

$$3*6*5*4*3=1080$$

Examples taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

Usual Patterns

- Orderings/Permutations
  - Application of the product rule
  - ***Distinguishable*** objects may be arranged in various different orders
    - **Note: Remeber D'Alembert's mistake !**
- Definition of Permutation
  - on n elements uses a $2 \times n$ matrix, where the first row is $(123 \ldots n)$ and the second row is the new arrangement. For example, the permutation which switches elements 1 and 2 and fixes 3 would be written as

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}.$$

Usual Patterns

- Counting permutations:
  - How to order n different objects?.

$$\{o_1, o_2, \cdots, o_n\}$$

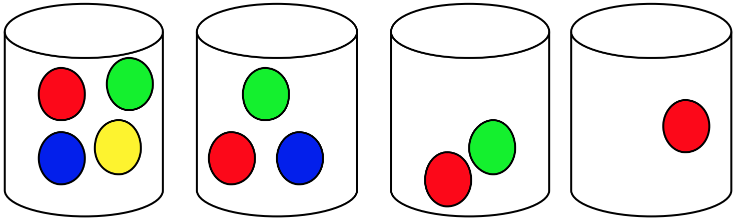
$$\{o_1, o_2, \cdots, o_n\} - \{o_i\}$$

$$\downarrow \qquad \ddots$$

$$\downarrow$$

$$\underbrace{X}_{n}\ \underbrace{X}_{n-1}\ \underbrace{X}_{n-2} \cdots \underbrace{X}_{2}\ \underbrace{X}_{1}$$

$$n(n-1)(n-2)\cdots 1 = n!$$



Usual Patterns

- Counting permutations/Comments:
  - How to order n different objects in r possitions?

$$\{o_1, o_2, \cdots, o_n\}$$

$$\{o_1, o_2, \cdots, o_n\} - \{o_i\}$$

$$\downarrow \qquad \ddots$$

$$\downarrow$$

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

$(n)_r$

$$\underbrace{X}_{n}\ \underbrace{X}_{n-1}\ \underbrace{X}_{n-2} \cdots \underbrace{X}_{n-(r-2)}\ \underbrace{X}_{n-(r-1)}$$

Is there a relation with the indistiguishable case?

Usual Patterns

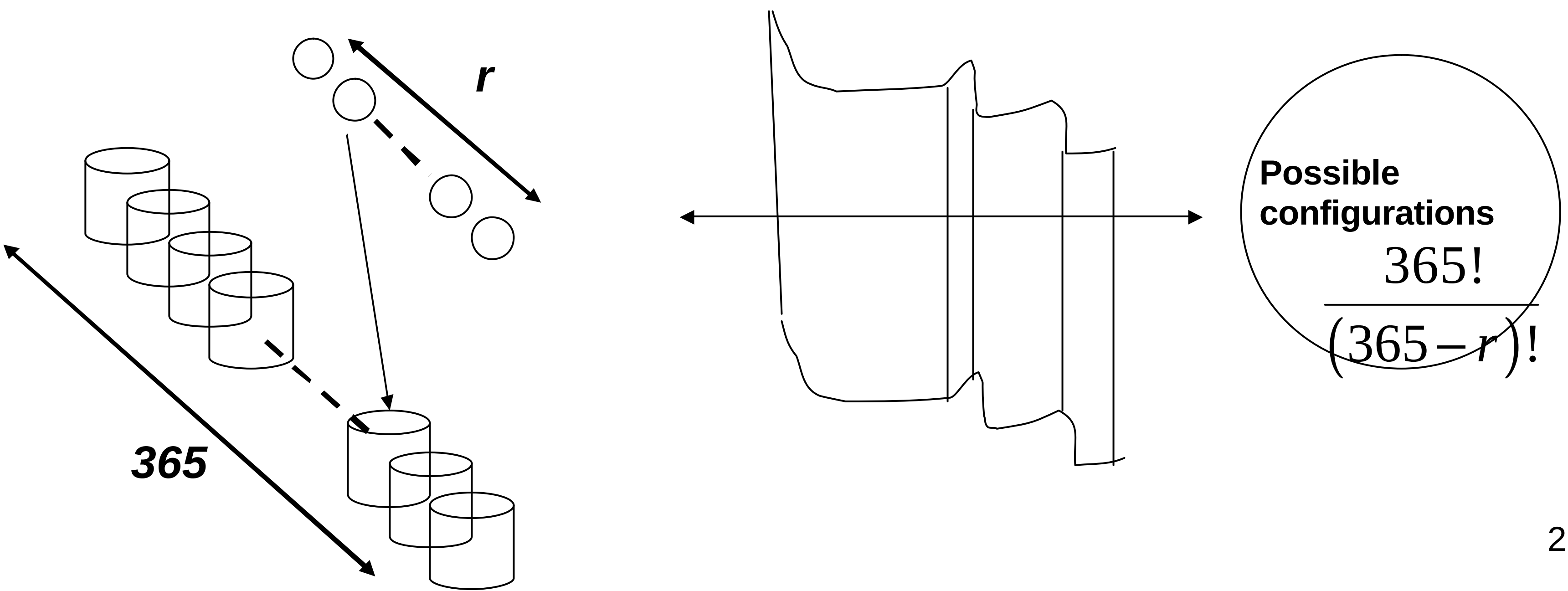
- Counting permutations:
  - Example: Possible noncoincident birthdays of r persons.

$$\underbrace{365}_{n}\underbrace{364}_{n-1}\underbrace{363}_{n-2}\cdots\underbrace{365-(r-2)}_{n-(r-2)}\underbrace{365-(r-1)}_{n-(r-1)}$$

  - Note: This example will be used for the coincidence problem, and for deriving the Poison pdf

Usual Patterns

- Counting permutations <u>*without*</u> repetition:
  - Another way of setting the problem.



$$\frac{365!}{(365-r)!}$$

Usual Patterns

- Counting permutations **<u>with</u>** repetition:
  – How to order n different objects with repetition?.

$$\left\{o_1, o_2, \cdots, o_n\right\}$$

$$\left\{o_1, o_2, \cdots, o_n\right\}$$

$$\downarrow \qquad \ddots$$

$$\downarrow$$

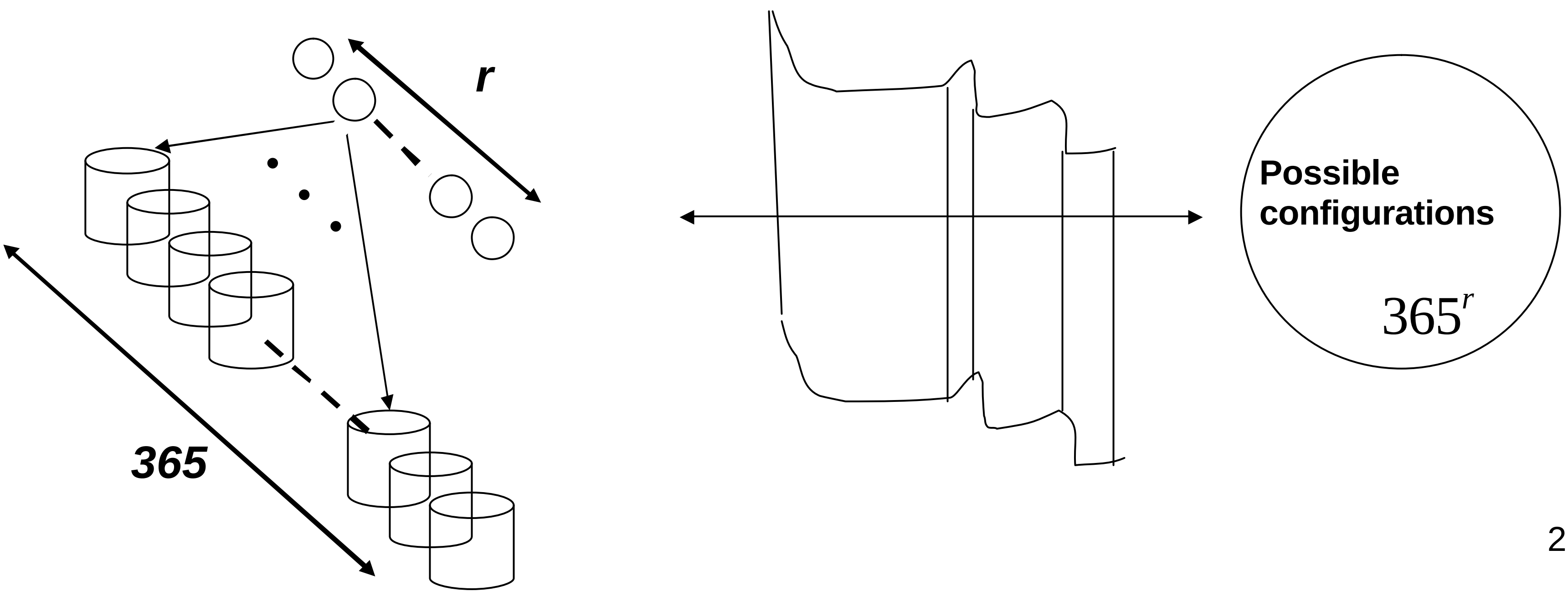
$$\underbrace{X}_{n}\ \underbrace{X}_{n}\ \underbrace{X}_{n} \cdots \underbrace{X}_{n}\ \underbrace{X}_{n}$$

$$\underbrace{nnn\cdots n}_{r} = n^r$$

http://www.kimbellart.org/database/images/AP1987_06.gif

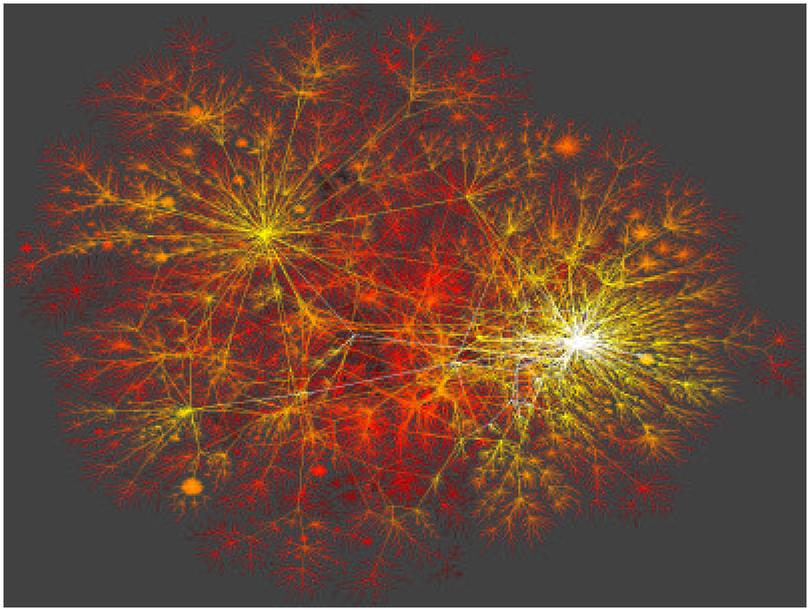
Usual Patterns

- Counting permutations with repetition:
  - Another way of setting the problem.



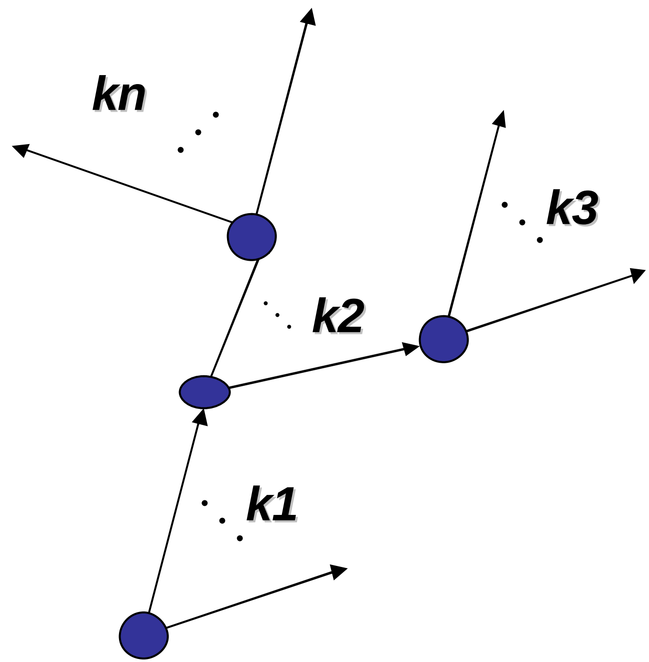
Usual Patterns

- Internet's Diameter:



- **Mean number of links/node=** $\bar{k}$
- **Number of nodes=N**

$$\bar{k} = \sum_{i=1}^{N} k_i$$



$$\bar{k}^{d} = N$$

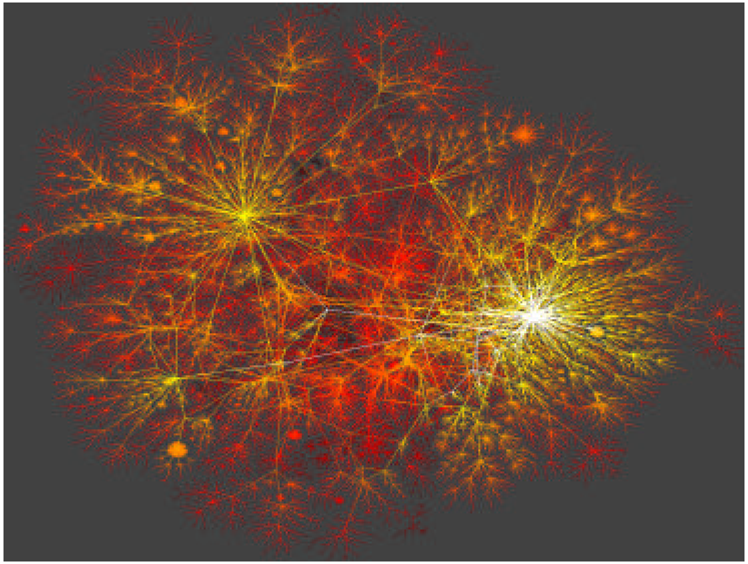
$$d = \frac{\log N}{\log \bar{k}} \approx 19$$

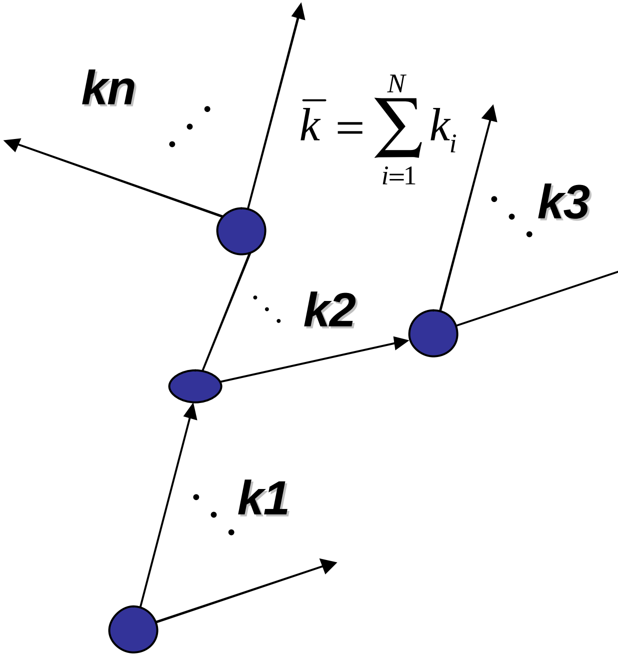
Usual Patterns

- Internet's Diameter:

•**Mean number of links/node=** $\bar{k}$

•**Number of nodes=N**





$$\bar{k}^{d} = N \qquad d = \frac{\log N}{\log \bar{k}} \approx 19$$

But $10^{16} >> 10^{9}$, what does it mean?

Maximum path from one node to the other.

Usual Patterns

– What does $d = \frac{\log N}{\log \bar{k}} \approx 19$ mean?

– Number of atoms in the universe is $N = 10^{70}$

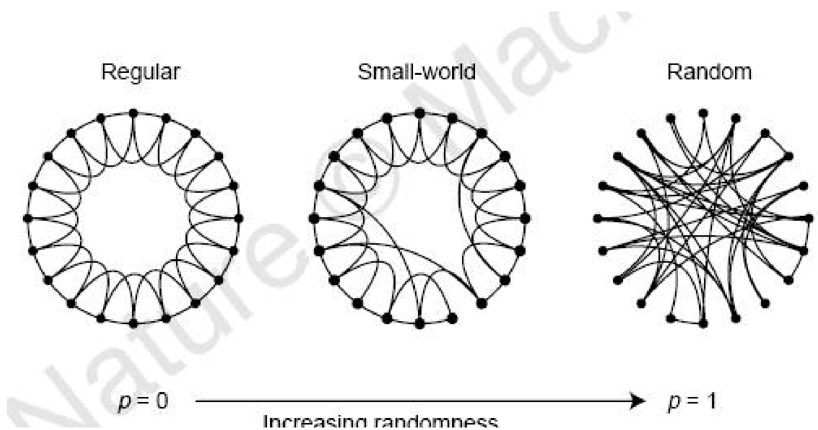
– The logarithm gives $d = \frac{\log_{10} N}{\log_{10} 10} = 70$

- Notice: the logarithm gives the number of decimal places needed to write the number.
  - It is a description of the number

•**Mean number of links/node=** $\bar{k}$
•**Number of nodes=N**

Usual Patterns

- Six Degrees
  - **Six degrees of separation** is the hypothesis that anyone on Earth can be connected to any other person on the planet through a chain of acquaintances with no more than five intermediaries.

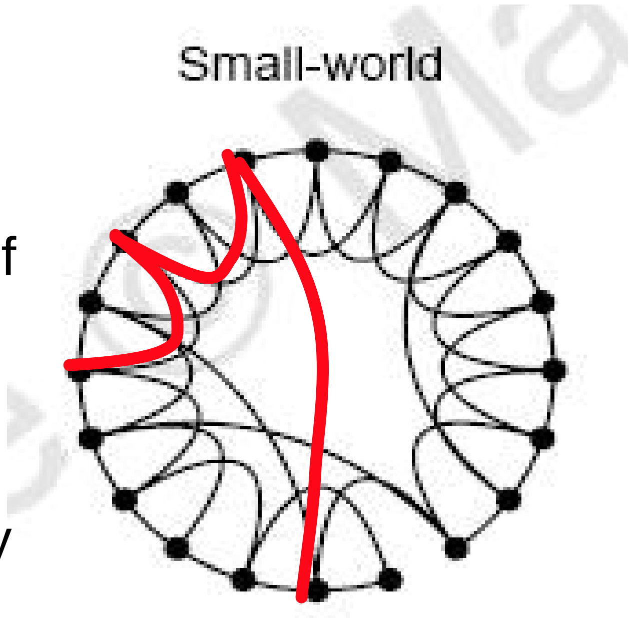


The Collective Dynamics of ‘Small-World’ Networks

D. J. Watts and S. H. Strogatz Nature, vol. 393, no. 6684, pp 440-2. (1998)

Usual Patterns

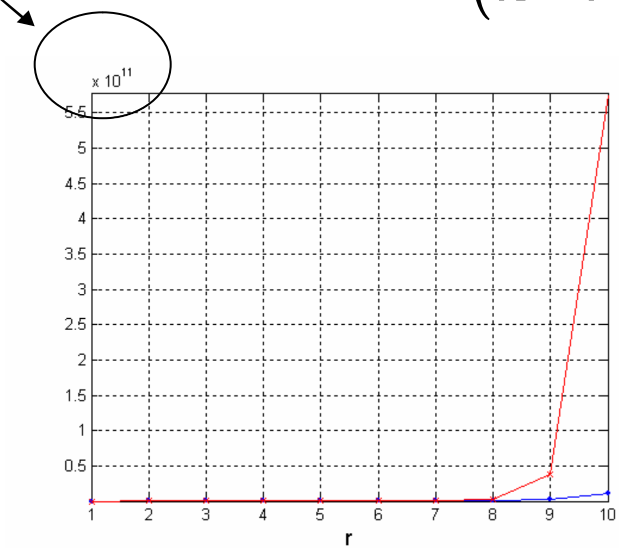
- Six Degrees/Example
  - How do you get from Muhammad Ali to Bob Dylan?
    - **1:** Muhammad Ali's manager for most of his career was Angelo Dundee.
    - **2:** Angelo Dundee also managed Heavyweight Champion Jimmy Ellis.
    - **3:** In 1964, Jimmy Ellis was defeated by Rubin "Hurricane" Carter.
    - **4:** The song *Hurricane* was written about Rubin Carter and appears on Bob Dylan's album *Desire*.

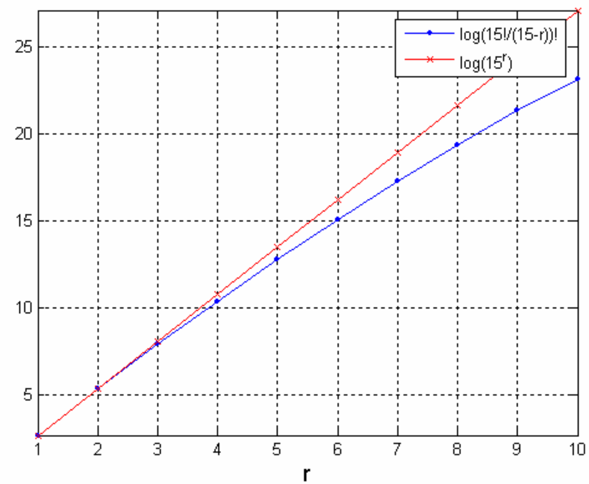


http://www.coudal.com/dylan/

Usual Patterns

- Comparison:
  - Number of orderings n different objects selected r times **with** and **without** repetition ?
    - n^r vs. $\dfrac{n!}{(n-r)!}$





Usual Patterns

- Counting permutations/ **indistinguishability**:
  - **Some** indistinguishable objects.
  - **n** objects each with $\left\{ n_1, n_2, \cdots, n_n \right\}$ elements
  - Notation: $\left\{ o_k^j \right\}$ Object k of class j
  - Notice that for a given permutation:

$$\left\{ X, o_1^j, o_2^j, X \cdots, X, o_{n_j}^j, X \right\} \rightarrow \left\{ X, o^j, o^j, X \cdots, X, o^j, X \right\}$$

$$\uparrow \quad \uparrow \qquad\qquad \uparrow \quad \textit{Fixed}$$

    - The $\left\{ n_j \right\}$ permutations are equivalent, therefore have to be discounted

Usual Patterns

- Counting permutations Some indistinguishable objects :

$$\left.\begin{matrix}\left\{X, o_1^j, o_2^j, X \cdots, X, o_3^j, X\right\} \\ \left\{X, o_1^j, o_3^j, X \cdots, X, o_2^j, X\right\} \\ \left\{X, o_2^j, o_1^j, X \cdots, X, o_3^j, X\right\} \\ \left\{X, o_2^j, o_3^j, X \cdots, X, o_1^j, X\right\} \\ \left\{X, o_3^j, o_1^j, X \cdots, X, o_2^j, X\right\} \\ \left\{X, o_3^j, o_2^j, X \cdots, X, o_1^j, X\right\}\end{matrix}\right\} \rightarrow \left\{X, o^j, o^j, X \cdots, X, o^j, X\right\}$$

All the $n_j!$ are different sets are equivalent to one.

$$Different\ Permutations \rightarrow \frac{n!}{n_j!}$$

Usual Patterns

- Counting permutations:
  - General Formula

$$\frac{n!}{n_1!n_2!\cdots n_k!} \quad n = n_1 + n_2 + \cdots + n_k$$

  - Different ways to order a collection of n objects with k subsets of indistinguishable objects.

Usual Patterns

- Counting permutations:
  - Example: With the simbols {a,a,a,b,b,c,c,d} how many words can be constructed?
  - Note that the same word can be constructed in different ways.

{a,a, b,c,a,b,c,d}

{a,a, b,c,a,b,c,d}

  - The number of words is: $$\frac{8!}{3!2!2!1!} = 1680$$

Examples taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

Usual Patterns

- Models of the equation
  - Thermodynamics: $\{n_j\}$ particles in a given state. How many different realizations can we have?
  - Language Models
  - Derivation of the Entropy (Thermodynamics/Information Theory)

$$(n_1, n_2, \ldots, n_k)! = \frac{(n_1 + n_2 + \ldots + n_k)!}{n_1 !\, n_2 ! \cdots n_k !}$$

Usual Patterns

- multinomial example (Rao, 1973)

There are n=197 animals classified into one of 4 categories:
$(y_1, y_2, y_3, y_4) = (125, 18, 20, 34)$

Number of different orderings is $\dfrac{n!}{y_1!\,y_2!\,y_3!\,y_4!}$

{a,a, b,c,a,b,c,d}

{a,a, b,c,a,b,c,d}

http://en.wikipedia.org/wiki/Image:Jan_Bruegel_d._%C3%84._003.jpg

Rao, C. R. (1973). Linear statistical inference and its applications. Wiley.

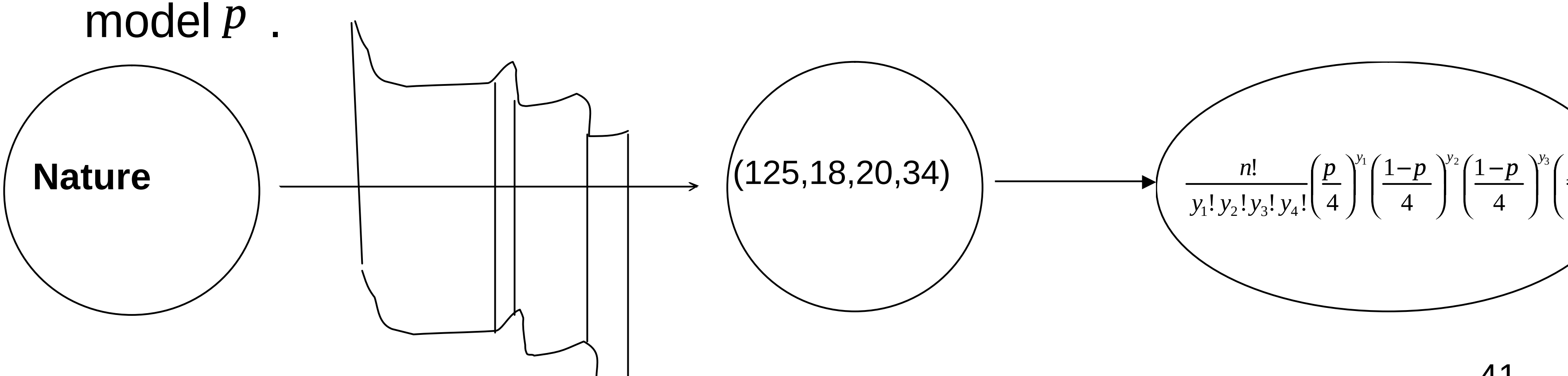
Usual Patterns

- multinomial example (Rao, 1973)

There are n=197 animals classified into one of 4 categories: $(y_1,y_2,y_3,y_4)=(125,18,20,34)$

Number of different orderings is $\dfrac{n!}{y_1!\,y_2!\,y_3!\,y_4!}$

Objective of the paper: test if the observations relate to the probability model p .



Rao, C. R. (1973). Linear statistical inference and its applications. Wiley.

Usual Patterns

- Language Models

Does a text correspond to an author?

Apriori probability of using a word: $\{p_1, p_2, \cdots, p_n\}$

A text has the word frequencies: $\{n_1, n_2, \cdots, n_n\}$

Number of different books with the same word frequenc¡es

$$\frac{n!}{n_1!n_2!\cdots n_n!}$$

Note: Does not take into account the probability of a word.

Usual Patterns

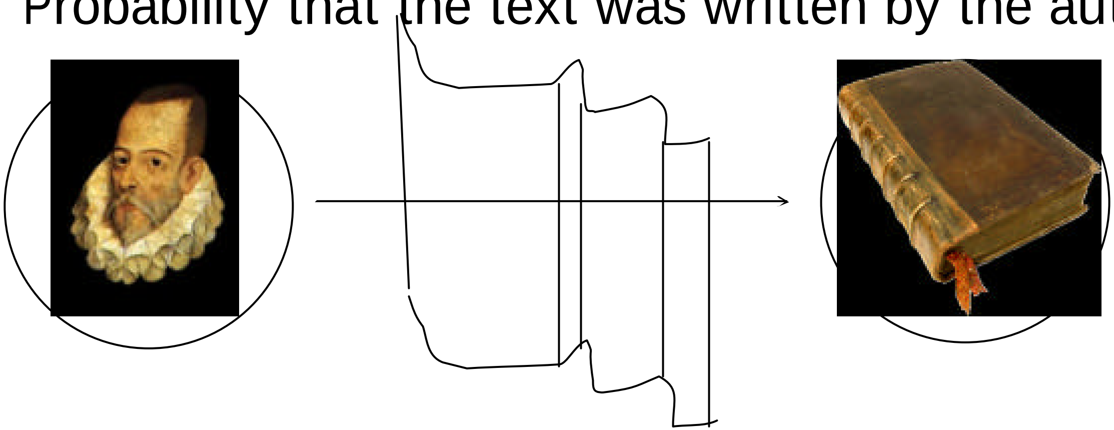
- Language Models

Does a text correspond to an author?

Apriori probability of using a word: $\{p_1, p_2, \cdots, p_n\}$

A text has the word frequencies: $\{n_1, n_2, \cdots, n_n\}$

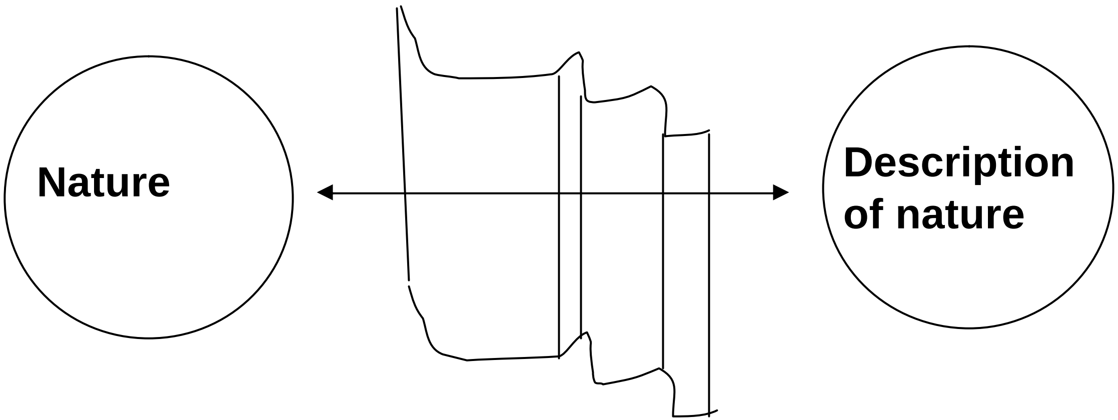
Probability that the text was written by the author



$$\frac{n!}{n_1!n_2!\cdots n_n!} p_1^{n_1} p_2^{n_2} \cdots p$$

Meanings of model

- Engineering vs. Mathematics/logic



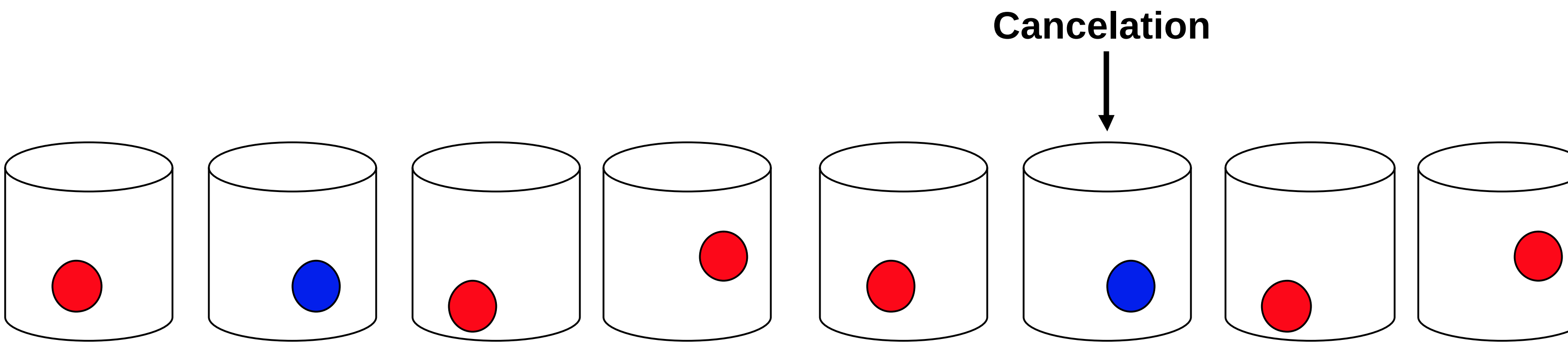
http://humanum.arts.cuhk.edu.hk/humftp/Fine_Arts/Gallery/matisse/matisse.jpg

Usual Patterns

- Subsets:
  - Given a set of ***n*** elements, how many subsets of cardinality ***r*** can be made?
    - Examples: How many possibilities when:
      - Throw a die ***n*** times and get ***r*** faces
      - Sell ***n*** sits in plane and have ***r*** cancelations
        - » Overbooking. D'alembert's mistake.

Usual Patterns

- Subsets:
  - How many possibilities when: Sell ***n*** sits in plane and have ***r*** cancelations
    - Notice 2 points:
      - Temporal order of booking is not important
      - Sets are ‘bookings’, not ‘cancelations’
        - » D’alembert’s mistake.



$$\{b_1, \quad b_2, \quad \cdots \quad \cdots \quad , b_{n-1}, \quad b_n\}$$

Usual Patterns

- n sits in plane and have r cancelations
  - Define a set of indices: $\left\{ i_1, i_2, \cdots, i_{n-r} \right\}$
  - Possible sets of bookings: $\left\{ b_{i_1}, b_{i_2}, \cdots, b_{i_{n-r}} \right\}$
    - Why ***(n-r)***? (orders?)
  - If each booking is different $n(n-1)(n-2)\cdots(n-r+1) = \dfrac{n!}{(n-r)!}$
  - But in a set order is not important
    - Is there a difference between the sets of cancelations:

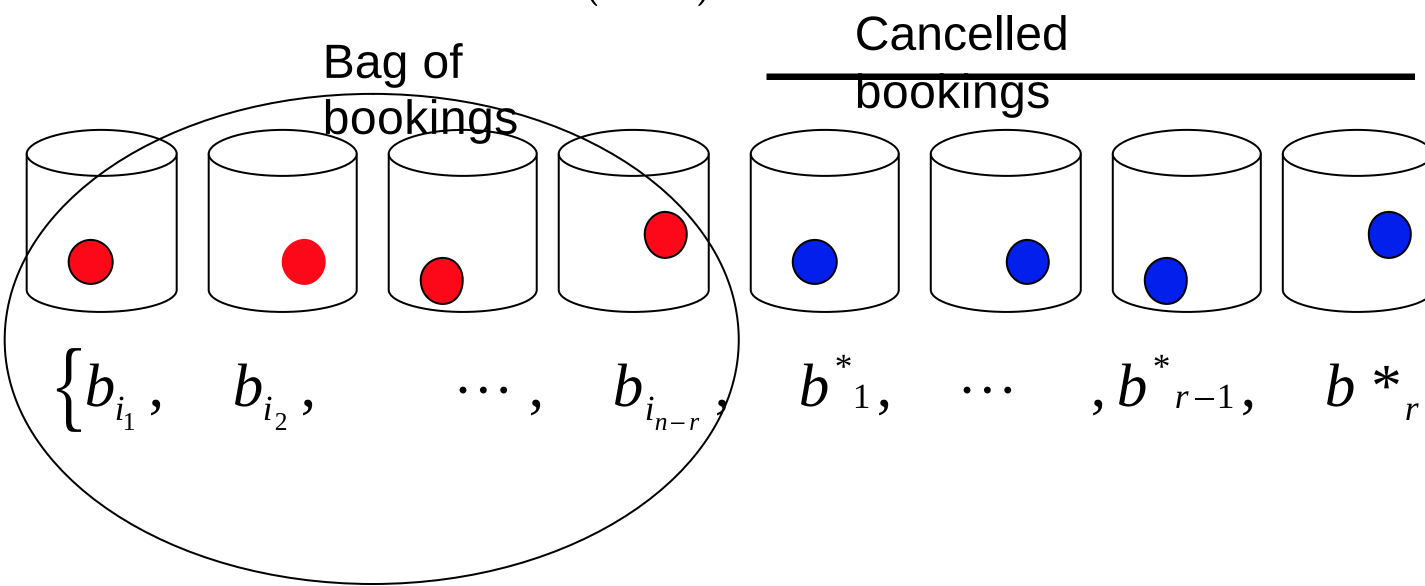
Usual Patterns

– ***n*** sits in plane and have ***r*** cancelations

- If each booking is different
- But in a set order is not important
  - – Is there a difference between the sets of cancelations?

$$n(n-1)(n-2)\cdots(n-r+1)=\frac{n!}{(n-r)!}$$

$$r(r-1)(r-2)\cdots 1 = r!$$

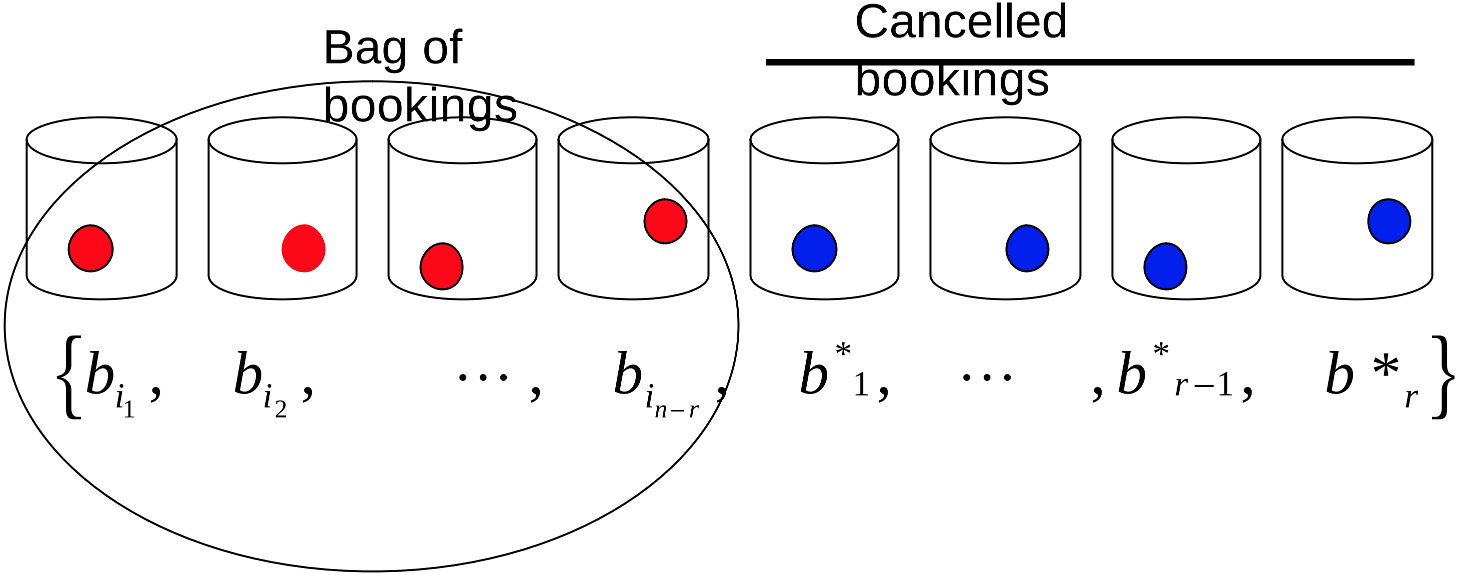


$$\{b_{i_1}, \quad b_{i_2}, \quad \cdots, \quad b_{i_{n-r}}, \quad b^*_1, \quad \cdots \quad , b^*_{r-1}, \quad b*_r\}$$

Usual Patterns

– ***n*** sits in plane and have ***r*** cancelations

$$n(n-1)(n-2)\cdots(n-r+1)=\frac{n!}{(n-r)!} \qquad r(r-1)(r-2)\cdots 1 = r!$$



$$\left\{b_{i_1},\quad b_{i_2},\quad \cdots,\quad b_{i_{n-r}},\quad b^*{}_1,\quad \cdots\quad ,b^*{}_{r-1},\quad b*_r\right\}$$

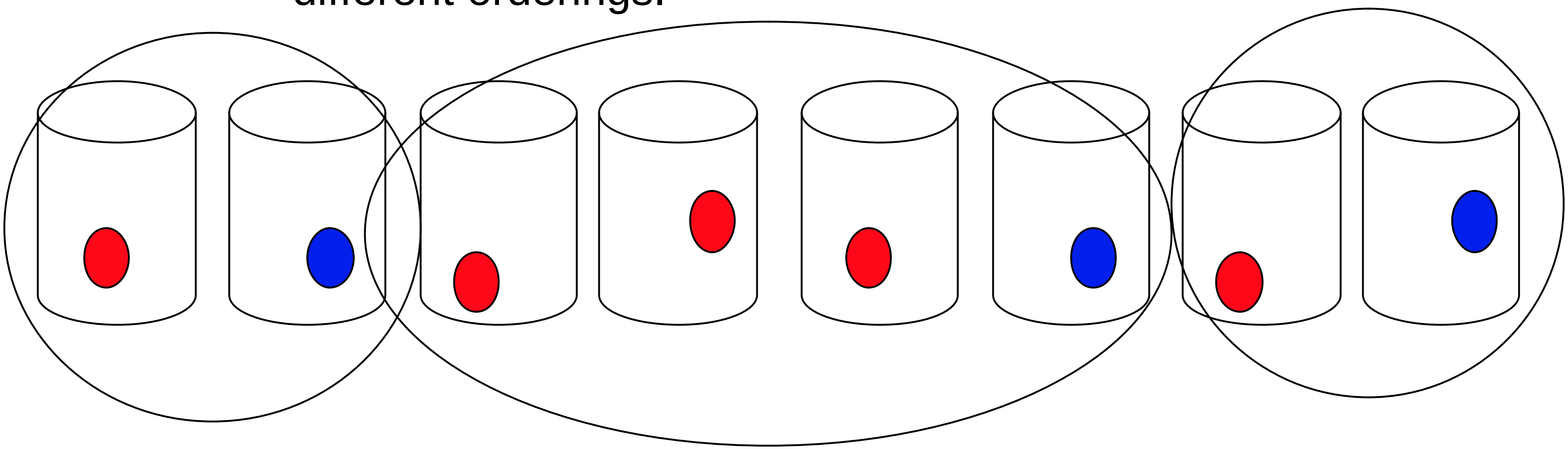
Discounting the orders of cancelations

$$Total=\frac{n!}{r!(n-r)!}$$

Usual Patterns

– ***n*** sits in plane and have ***r*** cancelations

- Another way of looking at the problem
  - Group the left neighbours of a cancelation, which gives ***r!*** different orderings.



$$\frac{n!}{(n-r)!}\frac{1}{r!}$$

Usual Patterns

- Subsets: General expression
  - How many possibilities when: Sell ***n*** sits in plane and have ***r*** cancelations

$$C(n,k)=\binom{n}{k}=\frac{n!}{(n-k)!\,k!}$$

    - Read ***n choose k***

Usual Patterns

- Properties of C(n,k):
  - Binomial Theorem

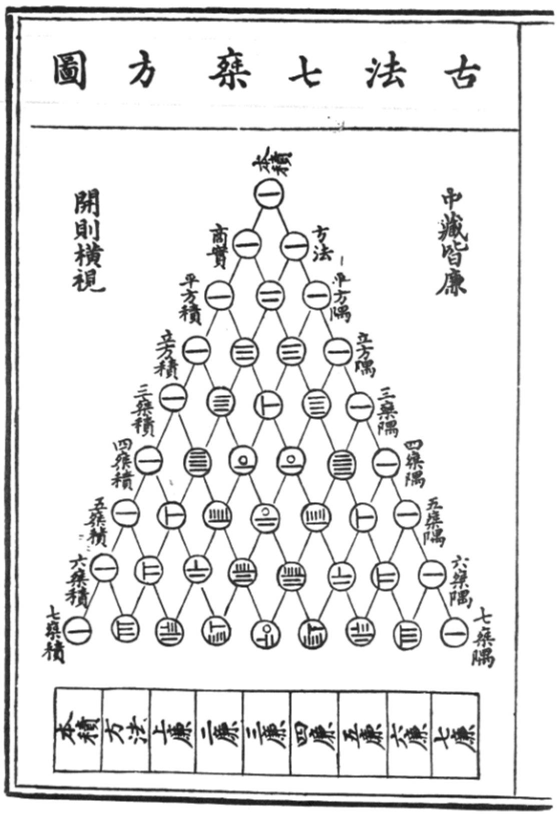
$$(a+b)^n \text{ for } n = 0, \ldots, 5$$

$$
\begin{array}{cc}
1 & (a+b)^0 \\
1a + 1b & (a+b)^1 \\
1a^2 + 2ab + 1b^2 & (a+b)^2 \\
1a^3 + 3a^2b + 3ab^2 + 1b^3 & (a+b)^3 \\
1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 & (a+b)^4 \\
1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5 & (a+b)^5
\end{array}
$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Usual Patterns

- Properties of C(n,k):
  - Pascal's triangle



C(n,k)	***n***
1	1
1 1	2
1 2 1	3
1 3 3 1	4
1 4 6 4 1	5
1 5 10 10 5 1	6

Usual Patterns

- Properties of C(n,k):
  - Recursive Computation

C(n,k)	***n***
1	1
1 1	2
1 2 1	3
1 3 3 1	4
1 4 6 4 1	5
1 5 10 10 5 1	6

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Usual Patterns

- Properties of C(n,k):
  - Binomial Theorem/Important consequence
    - Total number of subsets of a set.

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + b^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

$$\{aab, aba, baa\}$$

$$Number\ \ Subsets = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$$

Usual Patterns

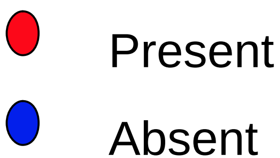
- Properties of C(n,k):
  - Total number of subsets of a set.
    - General Expression

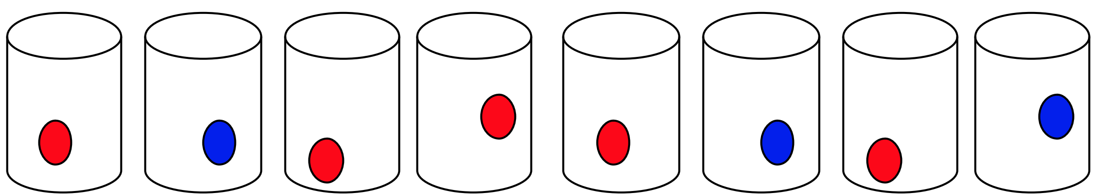
$$N = (1+1)^n = \sum_{k=1}^{n} \binom{n}{k} = 2^n$$

Usual Patterns

- Properties of C(n,k):
  - Total number of subsets of a set.
    - Another derivation

$$N = \underbrace{222\cdots2}_{n} = 2^n$$

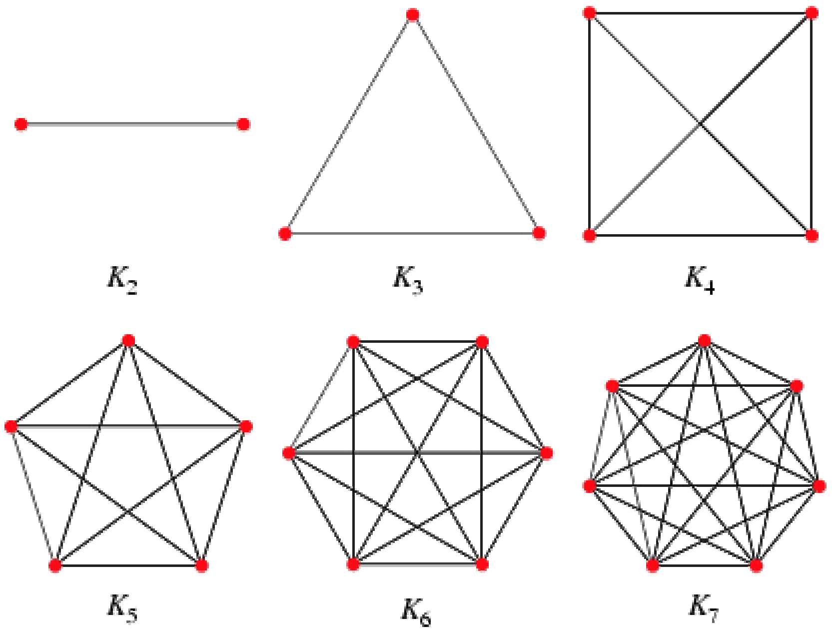




$$\{elem_1, \quad elem_2, \quad \cdots \quad \cdots \quad , elem_{n-1}, \quad elem_n\}$$

Usual Patterns

- Simple application:
  - Number of links in a graph



$$\binom{n}{2} = n\,(n-1) \Big/ 2$$

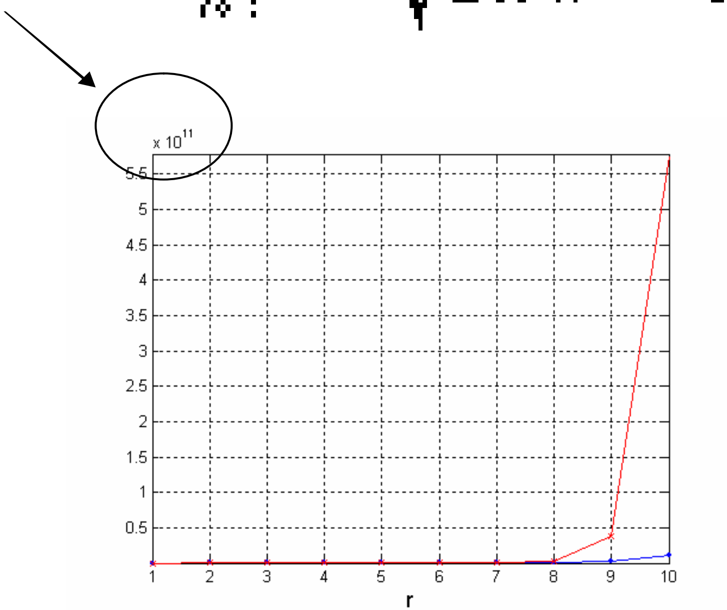
http://mathworld.wolfram.com/CompleteGraph.html

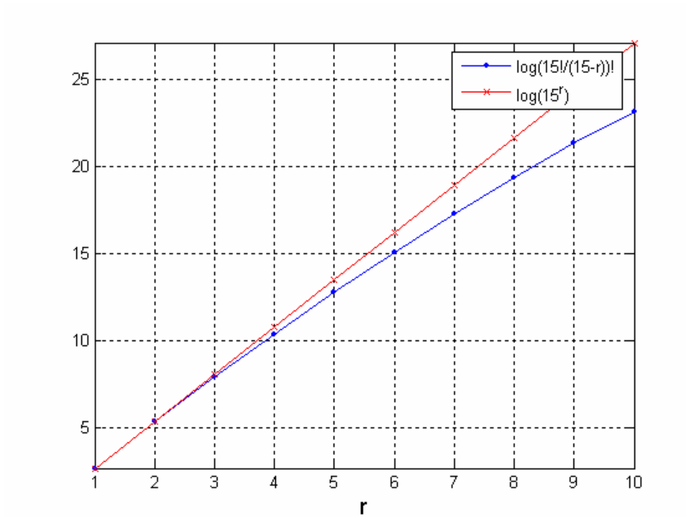
Stirling's Formula

- Justification of Stirling's Formula
  - Helps to deal and compute enourmous numbers

$$n! = \sqrt{2\pi}\, n^{n+1/2} e^{-n}$$

$$\frac{n!}{(n-r)!}$$





Stirling Formula

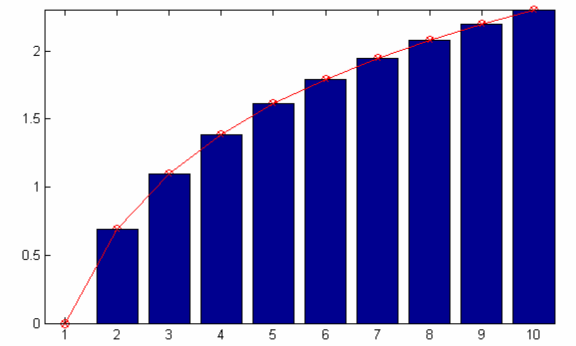
- Short derivation

$$\ln n! \;=\; \ln 1 + \ln 2 + \ldots + \ln n = \sum_{k=1}^{n} \ln k \approx \int_1^n \ln x \, dx$$

$$[x \ln x - x]_1^n = n \ln n - n + 1 \approx n \ln n - n.$$

$$\ln(n!) \approx n \ln(n) \quad n$$

$$n! \;=\; \sqrt{2\pi}\, n^{n+1/2}\, e^{-n}$$




```
t=1:10;plot(t,log(t),'r-o');hold on;bar(t,log(t));hold off;plot(t,log(t),'r-x');axis tight
```

1	1
2	2
3	6
4	24
5	120
6	720
7	5040
8	40320
9	362880
10	3628800

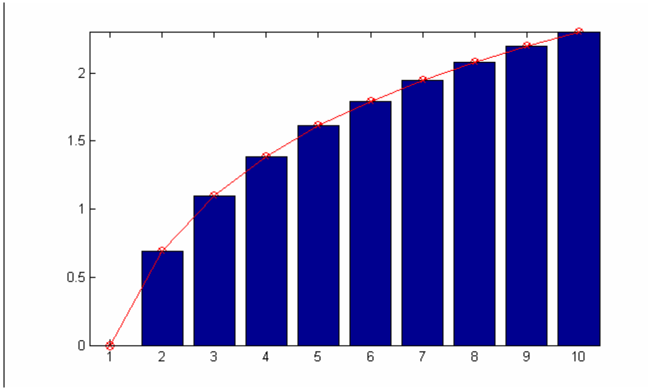
Weisstein, Eric W. "Stirling's Approximation." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/StirlingsApproximation.html

Stirling Formula

- Short derivation

$$n! = \sqrt{2\pi}\, n^{n+1/2}\, e^{-n}$$

n	n!	Stirling	Abs(Stirling -n!)	Stirling /n!
1	1	0.922137	0.077863	1.08444
2	2	1.919	0.0809956	1.04221
3	6	5.83621	0.16379	1.02806
4	24	23.5062	0.493825	1.02101
5	120	118.019	1.98083	1.01678
6	720	710.078	9.92182	1.01397
7	5040	4980.4	59.6042	1.01197
8	40320	39902.4	417.605	1.01047
9	362880	359537	3343.13	1.0093
10	3.63E+06	3.60E+06	30104.4	1.00837




```
t=1:10;plot(t,log(t),'r-o');hold on;bar(t,log(t));hold off;plot(t,log(t),'r-x');axis tight
```

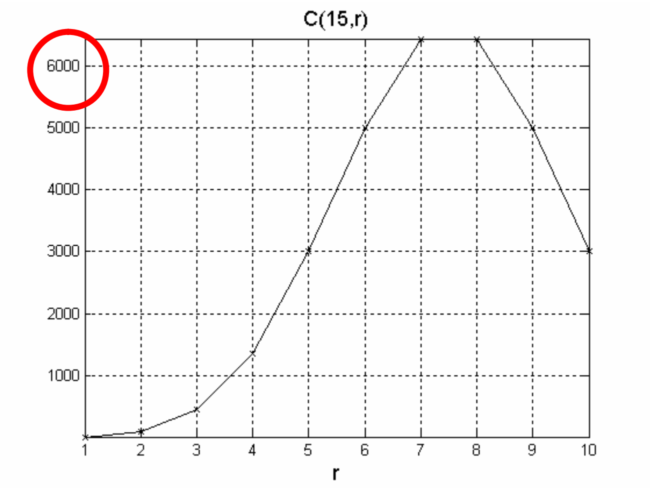
Important:

•Ratio not difference !

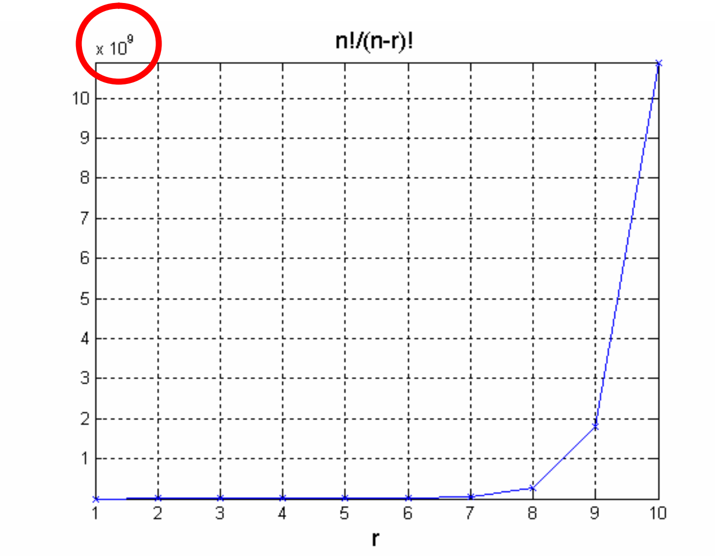
Weisstein, Eric W. "Stirling's Approximation." From *MathWorld*–A Wolfram Web Resource. http://mathworld.wolfram.com/StirlingsApproximation.html

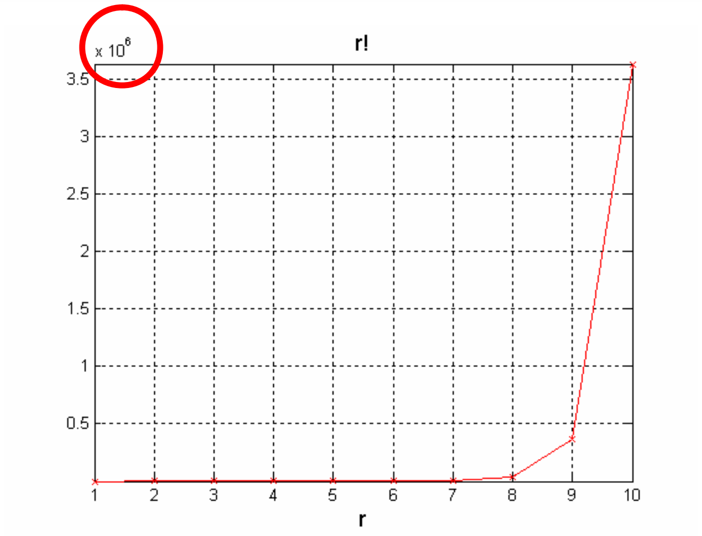
Stirling Formula

- Caveat
  - Division of ***enormous*** numbers



$$\frac{n!}{(n-r)!}\frac{1}{r!}$$





$$n! = \sqrt{2\pi}\, n^{n+1/2} e^{-n}$$

Exercices

- Ten persons select a paper each from an urn with ten papers. One paper has the prize. How many permutations are possible if the winner has made selection ‘i’.

Exercices

- Secretary problem (dowry, perfect husband, etc.)
  - You have 20 candidates and examine them sequentially.
  - How many possibilities do you have of selecting the best, if your strategy is to examine the first 10, and then select the best of the last 10.