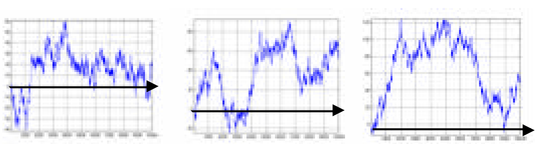
Random Variables

What is a random Variable?

- RV=***Experiment***+***Measure*** of interest
  - Roullette:
  - ***Experiment***
    - Roll the ball (rien ne va plus)
  - ***Measure***
    - A={A red has appeared}
    - B ={ An Odd number has appeared}
    - C ={ House winns}

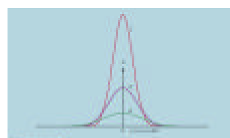
What is a random Variable?

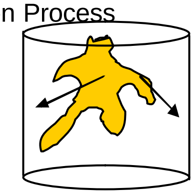
- ***Experiment***
  - Flip coins: Heads->+1,Tail->-1
- ***Measure***
  - A ={Fraction of time one is winning}
  - B ={Distance between zero crossings}
  - C ={Maximal distance to the zero point}

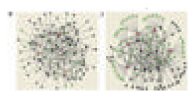


What is a random Variable?

- ***Experiment*** :Diffusion Process







Propagation of messages in a net

- ***Measure***
  - A ={Maximal distance as a function of time}
  - B={**Area/Perimeter**}

What is a random Variable?

- Definition:
- In a Probability Space (Ω,P), a Random Variable (rv), is a function:

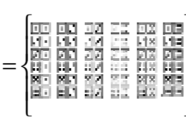
$$X:\Omega\rightarrow\Re$$

Example

- Two players (A,B) roll two dice
- Possible **random variables**:

$$X_A:\Omega\rightarrow\Re$$

$$X_A(i,j)=i/j \quad \text{for (i, j)} \in \Omega \qquad \Omega=$$



We would like to define a probability function

$P:\Omega\rightarrow\Re$ $\quad X_A(i,j)$

$P(X_A(i,j))$

0 1 ***P(.)***

Example

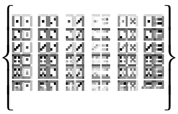
- Two players (A,B) roll two dice
- Possible random variables:

$X_{Win}: \Omega \rightarrow \Re$

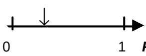
$$X_{Win}(i,j) = \begin{cases} 1 \text{ if } i > j \\ 0 \text{ if } i = j \\ -1 \text{ if } i < j \end{cases} \quad \text{for } (i,j) \in \Omega$$

$X_{Diff}: \Omega \rightarrow \Re$

$X_{Diff}(i,j) = i - j \quad \text{for } (i,j) \in \Omega$

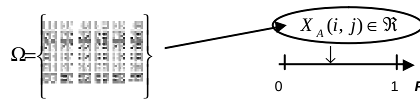
$\Omega =$ 

$X_{Win}(i,j)$ or $X_{Diff}(i,j)$

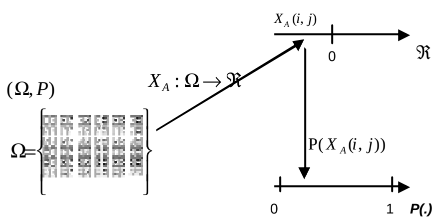


Objects that are needed

Name	*symbol*	*kind of object*
Sample Space	Ω	SET
Random Variable	$X_A: \Omega \rightarrow \Re$	FUNCTIONON A SET
Probability Space	(Ω, P)	SET + FUNCTIONON A SET
Prob.of a RV	$P(X_A(i,j))$	FUNCTIONON A $\Re$

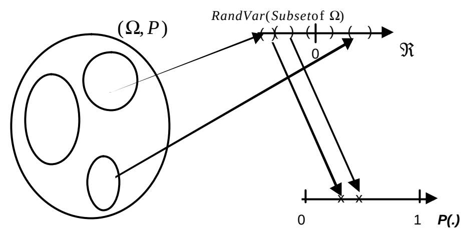


Relation between objects



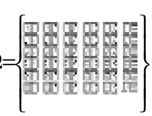
Probability of a Random Variable

- Definition: Set of probabilities associated to the values that a R.V. can take.



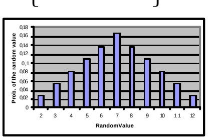
Example: Probability of a Random Variable

- A die is thrown twice

$\Omega =$ 

$X: \Omega \rightarrow \Re$

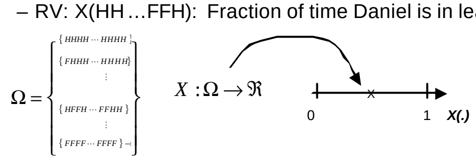
$X(i,j) = i + j \quad \text{for } (i,j) \in \Omega$

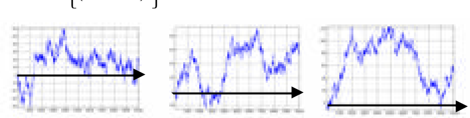


2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Example

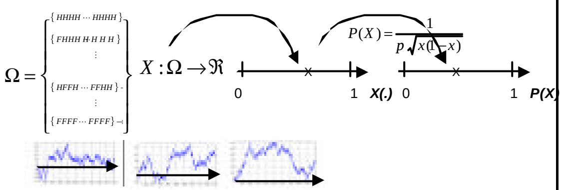
- Game of Daniel and Nicolas Bernouilli
  - RV: X(HH…FFH): Fraction of time Daniel is in lead.





Example

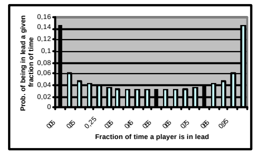
- Probability of X(*HH…FFH*):
  - We compute P(X(*HH…FFH*)) for each $A \in \Omega$



For the derivation of P(X) see Feller or Vélez

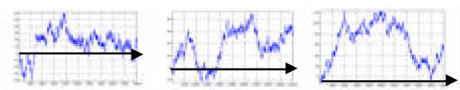
Example

- Probability of X(HH…FFH):
  - RV: Fraction of time Daniel is in lead.



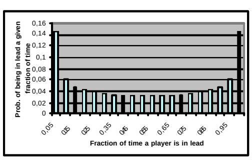
$X:\Omega \rightarrow \Re$

$$P(X) = \frac{1}{p\sqrt{x(1-x)}}$$



Example

- Probability of X(HH…FFH):
  - RV: Fraction of time Daniel is in lead.
    - Note that the distribution says that Daniel with high probability either wins of losses

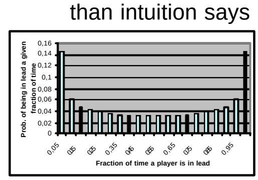


$X:\Omega \rightarrow \Re$

$$P(X) = \frac{1}{p\sqrt{x(1-x)}}$$

Example

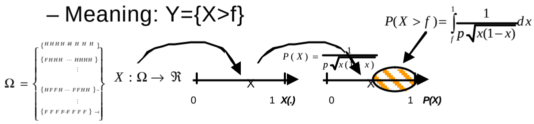
- Probability of X(HH…FFH):
  - Underlying mechanism.
  - In 20 throws the probability of a run of 5 consecutive heads or tails is much higher than intuition says



•Hot hand in basquetball
•Highly succesfull wall street trader
•Random numbers test.
Methods for Roullete
Shannon's coin game

Example

- New R.V. Y: Daniel is in lead more than a given fraction of time.
  - Meaning: Y={X>f}



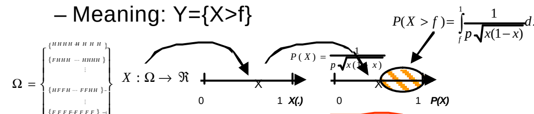
$Y:\Omega \rightarrow \Re$

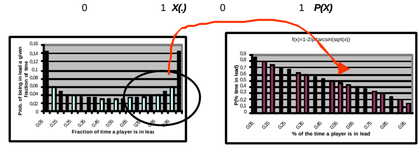
$\Omega \rightarrow X \rightarrow Y$

$$P(Y) = P(X > f) = \int_f^1 \frac{1}{p\sqrt{x(1-x)}}dx \approx 1 - \frac{2}{p}\arcsin(\sqrt{f})$$

Example

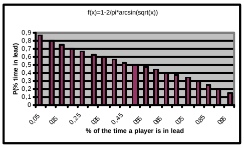
- New R.V. Y: Daniel is in lead more than a given fraction of time.
  - Meaning: Y={X>f}





Example

- New Rand.Var. *Y*: Daniel is in lead more than a given fraction of time.
  - Meaning: Y={X>f}



$Y:\Omega \to \Re$

$\Omega \to X \to Y$

$$P(Y)=P(X>f)=\int_f^1 \frac{1}{p\sqrt{x(1-x)}}dx \approx 1-\frac{2}{p}\arcsin(\sqrt{f})$$

Distribution of a Random Variable

- Definition:
  - The distribution function of a ***discrete*** random variable X is defined as:

$$F(x)=P(X\le x)=\sum_{\forall\, x_i\le x} P(X=x_i)$$

  - With

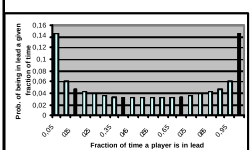
$$X:\Omega \to \Re$$
$$\Omega=\{A_1, A_2, \cdots A_n\}$$
$$\downarrow$$
$$X=\{x_1, x_2, \cdots x_m\}$$

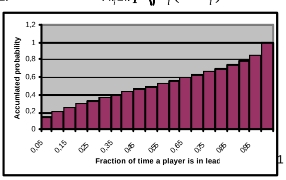
Example

- Distribution function of the Rand.Var.: Fraction of time Daniel is in lead

$X:\Omega \to \Re$

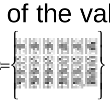
$$F(x)=P(X\le x)=\sum_{\forall\, x_i\le x} P(X=x_i)=\sum_{\forall\, x_i\le x}\frac{1}{p\sqrt{x_i(1-x_i)}}$$





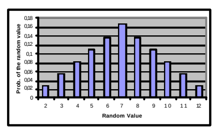
Example

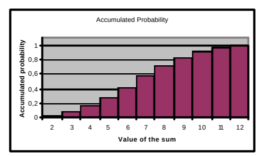
- Distribution function of the Rand.Var. : sum of the values when a die is thrown twice

$\Omega=$ 

$X:\Omega \to \Re$

$X(i,j)=i+j \quad \text{for } (i,j)\in\Omega$





$$F(x)=P(X\le x)=\sum_{\forall\, x_i\le x} P(X=x_i)$$

Example

- Distribution function of a Bernouilli trial until the first success, with probability of success ***p***.
  - Model of a Bernouilli trials:
    - Flipping a coin until a head appears.
    - Trying to get a connection until the service is given.
    - Booking a place in an airplane.
  - Probability of a success in ***n*** trials

$$\Omega \to N$$

$$P(X=n)=(1-p)^{n-1}p$$

Example

- Distribution function of a Bernouilli trial until the first success, with probability of success ***p***.

$$F(x)=P(X\le x)=\sum_{\forall n\le x} P(X=n)=\sum_{\forall n\le x}(1-p)^{n-1}p \underset{\text{Geometric Series}}{=}$$

$$=p\frac{1-(1-p)^n}{1-(1-p)}=1-(1-p)^n$$

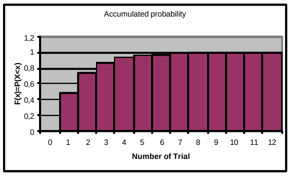
Note: Complementary of the event,={does appears in **none** of the n trials}

Example

- Distribution function of a Bernouilli trial until the first success, with probability of success ***p***.

$$F(x)=P(X\le x)=\sum_{\forall n\le x}P(X=n)=1-(1-p)^n$$

- Coin p=1/2

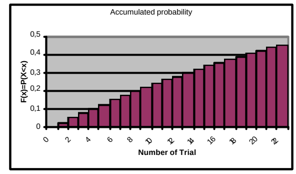


Example

- Distribution function of a Bernouilli trial until the first success, with probability of success ***p***.

$$F(x)=P(X\le x)=\sum_{\forall n\le x}P(X=n)=1-(1-p)^n$$

- Roullette p=1/37



Example: Propagation of a rumor

- In a village of **n** inhabitants, one person explains something to a second, who tells is to a third, etc.
- The first selects one in the **n-1** remaining. From then on, all the others select between **n-2** (i.e. excludes the previous, and the original)
  a) Distribution of the number of times that a rumor is transmited until it reaches again the source
  b) Distribution of the number of times that a rumor is transmited until someone get's it twice. (**Exercise**)

Taken from Cálculo de probabilidades I, R. Vélez & V. Hernández

Example: Propagation of a rumor

- Distribution of the number of times that a rumor is transmited until it reaches again the source.

$$A_0 \underset{\text{the rumor}}{\underset{\text{Generates}}{\rightarrow}} A_1 \underset{\text{different}}{\underset{\text{n-2}}{\rightarrow}} A_2 \underset{\text{different}}{\underset{\text{n-3}}{\rightarrow}} A_3 \underset{\text{different}}{\underset{\text{n-3}}{\rightarrow}} A_4 \cdots$$

  – A_j (j>2) cannot select the previous or de original
  – Note that A_1 and A_2 cannot select A_0

Example: Propagation of a rumor

- Distribution of the number of times that a rumor is transmited until it reaches again the origin.
- We define the random variable:
  – R: Number of times that the rumor is transmited until it returns to A_0
- Scheme of the solution:

$$P(R=r)=P(R>r-1)-P(R>r)$$

Example: Propagation of a rumor

- We will compute first $P(R>r)$
- Remember:

$$A_0 \underset{\text{the rumor}}{\underset{\text{Generates}}{\rightarrow}} A_1 \underset{\text{different}}{\underset{\text{n-2}}{\rightarrow}} A_2 \underset{\text{different}}{\underset{\text{n-3}}{\rightarrow}} A_3 \underset{\text{different}}{\underset{\text{n-3}}{\rightarrow}} A_4 \cdots \underset{\text{different}}{\underset{\text{n-3}}{\rightarrow}} A_r \underset{\text{different}}{\underset{\text{n-3}}{\rightarrow}} A_{r+1} \underset{\text{different}}{\underset{\text{n-2}}{\rightarrow}} \cdots$$

  – A_j (j>2) cannot select the previous or de original
  – Note that A_1 and A_2 cannot select A_0
    - A_1 and A_2 do not count. A_1 still has to talk to someone
- Event {R>r}

$$\{R>r\}=\{A_3,A_4\cdots,A_r\ne A_0\}AND\{\text{Any other sequence}\}$$

$$P(R>r)=\left(\frac{n-3}{n-2}\right)^{r-2}$$

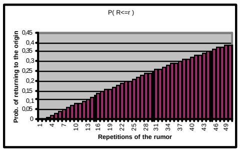
Example: Propagation of a rumor

- Distribution function of R

$$F(r) = P(R \le r) = 1 - P(R > r)$$

$$F(r) = \begin{cases} 0 & \text{if } r < 3 \\ 1 - \left(\dfrac{n-3}{n-2}\right)^{r-2} & \text{if } r \ge 3 \end{cases}$$

In a department
with **n=100**



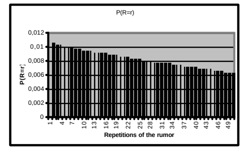
Example: Propagation of a rumor

- Probability of a given ***r***

$$P(R = r) = P(R > r-1) - P(R > r)$$

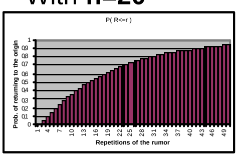
$$P(R = r) = \left(\frac{n-3}{n-2}\right)^{r-3} - \left(\frac{n-3}{n-2}\right)^{r-2} = \left(\frac{n-3}{n-2}\right)^{r-3} \frac{1}{n-2}$$

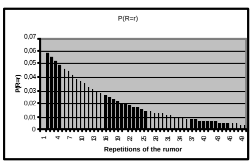
In a department
with **n=100**



Example: Propagation of a rumor

- With **n=20**





- Note that there is the possibility of *loops*