Conditioned probability

\[ P(2) = \frac{11}{36} \]
\[ P(2/\text{Second die is 1}) = \frac{1}{36} \]
\[ P(2/\text{Second die is 3}) = \frac{1}{36} \]

Intuition

- Time/Actions change the sample space.

\[ P(k) = 0.0084 \]
\[ P(k/y) = 0 \]
\[ P(k/a) = 0.001 \]

Formal definition

- Prob. of A conditioned to B is defined as

\[ P(A/B) = \frac{P(A \cap B)}{P(B)} \]

\[ \text{Pr} = \frac{\text{Count of ways for a result}}{\text{Count of all possible results}} \]

An Example: The sister problem

- You knock at the door of a family with two children, and a girl opens the door.
- Which is the probability that the other child is a boy?
- \( \Omega = \{(B, B), (G, B), (B, G), (G, G)\} \)
- \( A = \{\text{one child is boy}\} \)
- \( B = \{\text{one child is girl}\} \)

\[ P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2}{3} \]
**An Example: The sister problem**

Possible solutions

\[
\Omega_1 = \{B, G\}, \{G, B\}
\]

\[
P(A/B) = 1/2
\]

\[
\Omega_2 = \{(B_1, B_2), (G_1, B), (B_2, G_1), (G, G_2)\}
\]

\[
P(A/B) = 2/3
\]

**Possible Worlds**

(Collapsed) \(\Omega = \{B, B, G, G\}\)

(Collapsed) \(\Omega = \{B, B, G, G\}\)

There is only one house, and you knock the door

There are 4 houses, you choose one and knock the door

\[
\frac{2}{1}/\frac{1}{2} = \frac{2}{2} = \Omega
\]

\[
\frac{B}{A}/\frac{G}{B} = \Omega
\]


An Example: The prisoner’s dilemma
(in prob. different from the game theory)

- Chance tree/Decision tree

Judge decides | Guard says | Final prob.
---|---|---
1/3 | A free | 1/2
1/3 | B free | 1
1/3 | C free | 1

Asumptions on \{O_1, O_2, O_3, O_4\}

- Asumption that the original events have equal probability

Asumptions on \{O_1, O_2, O_3, O_4\}

- Asumption that the original events have equal probability

Real world | Model describing the Real World
---|---
\{O_1, O_2, O_3, O_4\} | \(P(O_1) = \frac{1}{6}\)
\(O_1\) | \(1/5\)
\(O_2\) | \(1/5\)
\(O_3\) | \(1/5\)

An Example: The Monty Hall dilemma

- It makes no difference whether the player switched doors or not. Each of the two remaining unopened doors had a \(\frac{1}{2}\) prob. of concealing the automobile.

- If it is not in the opened door, it is behind either, so therefore the prob. is of \(\frac{1}{2}\)

- **Caveat** is it really random?, Note that the host has choosen with knowledge.

- Is the the same sample space?
An Example: The Monty Hall dilemma

- Chance tree/Decision tree

The car is behind host opens Final prob.

1/3 \(\rightarrow\) door 1 \(\rightarrow\) \(1/2\) \(\rightarrow\) door 2 \(\rightarrow\) \(1/6\)

1/3 \(\rightarrow\) door 2 \(\rightarrow\) 1 \(\rightarrow\) door 3 \(\rightarrow\) \(1/3\)

1/3 \(\rightarrow\) door 3 \(\rightarrow\) 1 \(\rightarrow\) door 2 \(\rightarrow\) \(1/3\)

Taken from: Understanding Prob. Chance rules in everyday life. H. Tijms

An Example: The Monty Hall dilemma

- Probabilities

The car is behind host opens Final prob.

1/3 \(\rightarrow\) door 1 \(\rightarrow\) \(1/2\) \(\rightarrow\) door 2 \(\rightarrow\) \(1/6\)

1/3 \(\rightarrow\) door 2 \(\rightarrow\) 1 \(\rightarrow\) door 3 \(\rightarrow\) \(1/3\)

1/3 \(\rightarrow\) door 3 \(\rightarrow\) 1 \(\rightarrow\) door 2 \(\rightarrow\) \(1/3\)

Taken from: Understanding Prob. Chance rules in everyday life. H. Tijms

An Example: The Monty Hall dilemma

- Note that conditioning changes the sample space

\(\Omega = \{\text{door 1}, \text{door 3}\}\)

\(\Omega_{\text{prob}} = \{\text{door 1}, \text{door 2}, \text{door 3}\}\)

The car is behind host opens Final prob.

1/3 \(\rightarrow\) door 1 \(\rightarrow\) \(1/2\) \(\rightarrow\) door 2 \(\rightarrow\) \(1/6\)

1/3 \(\rightarrow\) door 2 \(\rightarrow\) 1 \(\rightarrow\) door 3 \(\rightarrow\) \(1/3\)

1/3 \(\rightarrow\) door 3 \(\rightarrow\) 1 \(\rightarrow\) door 2 \(\rightarrow\) \(1/3\)

Taken from: Understanding Prob. Chance rules in everyday life. H. Tijms

An Example: The Monty Hall dilemma

- Note that conditioning changes the sample space

\(\Omega_{\text{prob}} = \{\text{door 1}, \text{door 2}, \text{door 3}\}\)

\(\Omega = \{\text{door 1}, \text{door 3}\}\)

\(\Omega_{\text{prob}} = \{\text{door 1}, \text{door 2}, \text{door 3}\}\)

\(\Omega_{\text{prob}} = \{\text{door 1}, \text{door 2}\}\)

\(\Omega_{\text{prob}} = \{\text{door 1}, \text{door 3}\}\)

\(\Omega_{\text{prob}} = \{\text{door 1}, \text{door 2}, \text{door 3}\}\)

\(\Omega_{\text{prob}} = \{\text{door 1}, \text{door 3}\}\)

The car is behind host opens Final prob.

1/3 \(\rightarrow\) door 1 \(\rightarrow\) \(1/2\) \(\rightarrow\) door 2 \(\rightarrow\) \(1/6\)

1/3 \(\rightarrow\) door 2 \(\rightarrow\) 1 \(\rightarrow\) door 3 \(\rightarrow\) \(1/3\)

1/3 \(\rightarrow\) door 3 \(\rightarrow\) 1 \(\rightarrow\) door 2 \(\rightarrow\) \(1/3\)

Taken from: Understanding Prob. Chance rules in everyday life. H. Tijms

Computation of intersection probability

- Temporal structure: B takes place and afterwards A

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(A \cap B) = P(B)P(A \mid B) \]

- Generalization for a sample space

\(\Omega = \{A_1, A_2, \ldots A_n\}\)

\[ P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2 \cap \cdots \cap A_n) = \]

\[ = P(A_1)P(A_2 \cap \cdots \cap A_n) = \]

\[ P(A \cap B) = P(A)P(B \mid A) \]
Computation of intersection probability

• Generalization for a sample space

\[ \Omega = \{ A_1, A_2, \ldots, A_n \} \]

\[ P(A_1 \cap A_2 \cap \cdots \cap A_n) = \]

\[ = P(A_1) P(A_2 / A_1) P(A_3 / A_1 \cap A_2) \cdots P(A_n / A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \]

\[ P(A_i \cap \cdots A_i) / A_i = \]

\[ = P(A_i / A_1) P(A_2 \cap \cdots \cap A_i / A_1 \cap A_i) \]

Computation of intersection probability

• Examples

– Birthday problem

\[ \{ A_i \} = \text{the day of the birthday of individual } i \]

\[ \text{Sample space } \Omega = \text{the days of all the others}. \]

\[ P(A_i / A_1) \rightarrow \text{Probability of the birth day of } A_i \text{ on day } A_j \]

Formation of sentence

\[ \{ F_{\text{slw}} \} \]

Example

• We can connect to two servers:
  – S1 has 2 high speed links and 1 slow link
  – S2 has 1 high speed link and 3 slow links

• We select one server at random

• Which is the probability of getting a slow link?

To be, or not to be— that is the question;

Whether 'tis nobler in the mind to suffer

The slings and arrows of outrageous fortune,

Or to take arms against a sea of troubles,

And by opposing end them? To die, to sleep

Microsoft® Encarta®
Example

• Sample space:

\[ \Omega = \{ S_1Fst, S_1Slw, S_2Fst, S_2Slw \} \]

- Event: Select Server and then type of link

\[ \frac{2}{3} \rightarrow \text{HighSpeed} \]
\[ \frac{1}{3} \rightarrow \text{LowSpeed} \]
\[ \frac{1}{2} \rightarrow S_1 \]
\[ \frac{3}{4} \rightarrow \text{HighSpeed} \]
\[ \frac{1}{4} \rightarrow \text{LowSpeed} \]

\[ \text{Prob}=\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{3}{4} = \frac{13}{24} \]

Example

• Probabilities:

\[ P(S_1) = \frac{1}{2} \]
\[ P(Slw / S_1) = \frac{1}{3} \]
\[ P(Slw / S_2) = \frac{3}{4} \]
\[ P(Slw) = P(S_1)P(Slw / S_1) + P(S_2)P(Slw / S_2) \]

Example

• Probabilities: Another way

\[ P(Slw) = P((Slw \cap S_1) \cup (Slw \cap S_2)) \]
\[ P(Slw) = P(Slw \cap S_1) + P(Slw \cap S_2) \]
\[ P(Slw) = P(S_1)P(Slw / S_1) + P(S_1)P(Slw / S_1) \]

Properties

• The universe conditioned to a given event has prob. one:

\[ P(\Omega / B) = \frac{P(\Omega \cap B)}{P(B)} = 1 \]

• Both prob. spaces will have the same properties

\[ (\Omega, P(O / B)) \quad (B, P(B / \Omega)) \]

Properties

• Total probabilities:

- Given a set of events \( S = \{ A_1, A_2, \ldots, A_n \} \) pairwise disjoint or mutually disjoint, such that

\[ B_i \cup B_j \cup \cdots \cup B_k = \Omega \]

- We have:

\[ P(A) = P(B_1)P(A / B_1) + P(B_2)P(A / B_2) + \cdots + P(B_n)P(A / B_n) \]

1 - MutuallyDisjoint: for all \((i, j)\) such that \( i \neq j \)

\[ B_i \cap B_j = 0 \]
Properties

• Total probabilities:
  
  - Proof:

\[
A = A \cap \Omega = A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = \]

\[
(A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n)
\]

\[
P(A) = P(A \cap B_1) \cup P(A \cap B_2) \cup \cdots \cup P(A \cap B_n)
\]

\[
\text{but} \quad P(A \cap B_j) = P(B_j)P(A \mid B_j)
\]

\[
P(A) = P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + \cdots + P(B_n)P(A \mid B_n)
\]

Example

• A Clinical analysis is used for the diagnosis of three illness, B_1, B_2, B_3.

• The proportion of people with a given illness is: 3%, 2%, 1%

• The analysis gives a positive result for:

  - B_1 → 85%
  - B_2 → 92%
  - B_3 → 78%
  - B_4 → 0.5%

• Compute the probability of a positive.

Example

• What would happen with P(A \mid B_1) = 0.1 ?

\[
P(A) = P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2) + P(B_3)P(A \mid B_3) + P(B_4)P(A \mid B_4)
\]

\[
= 0.03 \times 0.85 + 0.02 \times 0.92 + 0.01 \times 0.78 + 0.94 \times 0.1 = 0.146
\]

Why 14.6%? Only 6% should give positive!!!!!!

Bayes’s formula

• Intuitively not clear (?)

\[
P(A \mid B_j) = \frac{P(B_j)P(A \mid B_j)}{P(B_j)P(A \mid B_j) + P(B_{j'} \mid B_j) + \cdots + P(B_n \mid B_j)}
\]

• Context:
  
  - Solution to a problem of “inverse probability” was presented in the Essay Towards Solving a Problem in the Doctrine of Chances
  
  - Published Divine Benevolence, or an Attempt to Prove That the Principal End of the Divine Providence and Government is the Happiness of His Creatures (?)
  
  - End is synonym of purpose, aim.
Kinds of probability

- Probability of an observation
- Probability of the cause of the observation
- Probability of the estimate of the probability,

\[
p_{\text{white}} = P(\text{white observation} | \text{composition of urn})
\]

Bayes Formula

- Motivation:
  - We would like the probability of the causes that generate the observations.
  - Update our knowledge after the observations.

\[
p_{\text{white}} = P(A \text{ given observation} / \text{composition of urn})
\]

Bayes Formula

- Context of the Bayes formula:
  - Kant's knowledge theory

\[
p_{\text{white}} = P(A \text{ given observation} / \text{composition of urn})
\]

Bayes Formula

- Laplace proposes taking into account the probability of the probability:

\[
p_{\text{white}} = P(A \text{ given observation} / \text{composition of urn})
\]

Bayes's formula

- Given a set of events \( S = \{B_1, B_2, \ldots, B_n\} \) pairwise disjoint or mutually disjoint¹, such that

\( B_1 \cup B_2 \cup \cdots \cup B_n = \Omega \)

- We have:

\[
P(B_j / A) = \frac{P(B_j \cap A)}{P(A)} = \frac{P(B_j)P(A / B_j)}{P(A)}
\]

\[P(B_j / A) = \frac{P(B_j)P(A / B_j)}{P(B_j)P(A / B_j) + P(B_j)P(A / B_j) + \cdots + P(B_j)P(A / B_j)}
\]

¹: Mutually Disjoint : for all \(i,j\) such that \(i \neq j\)

\(B_i \cap B_j = 0\)
Bayes’s formula

- Expectation
  1. anticipation of something happening: a confident belief or strong hope that a particular event will happen
  2. notion of something: a mental image of something expected, often compared to its reality (often used in the plural)
  3. expected standard: a standard of conduct or performance expected by or of somebody (often used in the plural)

Bayes’s formula and Utility function

- Idea of utility
  - Expectation is a subjective concept.
  - Instead of the probabilistic expectation, we could think in the ‘moral expectation’

Value = Chance of the thing \(X\) depending on the happening

\[
\text{Value}(B_i | A) = P(B_i | A) \cdot \text{Utility}(A | B_i)
\]

Application to the clinical analysis problem

- Probability of having the illness given a positive result \(A\).

\[
\begin{align*}
P(B_1 | A) & = \frac{P(B_1 | A) \cdot P(A | B_1)}{P(A)} = 0.03 \cdot 0.85 = 0.0265 \approx 0.026
\end{align*}
\]

- Meaning:

\[
\begin{align*}
P(B_1 / A) & \to 45% \\
P(B_2 / A) & \to 32% \\
P(B_3 / A) & \to 13% \\
P(B_4 / A) & \to 8.3% \quad \text{False positive}
\end{align*}
\]

Bayes and chance trees

- Urn has 5 white balls and 3 black. One ball is taken randomly and introduced in another with 2 white and 1 black.
- A ball white is taken from the second urn.
- What is the probability that the first ball was black?
Bayes and chance trees

- Urn has 5 white balls and 3 black. One ball is taken randomly and introduced in another with 2 white and 1 black.
- A ball white is taken from the second urn.
- What is the probability that the first ball was black?

\[
P(W_1/W') = \frac{P(W_1)P(W'/W)}{P(W_1)P(W'/W) + P(W)P(W'/W)} = \frac{5/8 \times 3/4}{5/8 \times 3/4 + 3/8 \times 1/2} = \frac{15/16}{1/2} = \frac{3}{4}
\]

Bayes and chance trees

- What is the probability that the first ball was black?

\[
P(W_1/W') = \frac{P(W_1)P(W'/W)}{P(W_1)P(W'/W) + P(W)P(W'/W)} = \frac{5/8 \times 3/4}{5/8 \times 3/4 + 3/8 \times 1/2} = \frac{15/16}{1/2} = \frac{3}{4}
\]

Bayes and “the prosecutor’s fallacy”.

- The prosecutor’s fallacy is the assertion that, because the story before the court is highly improbable, the defendant’s innocence is equally improbable.

- OJ Simpson:
  - Prob. Having a beating husband given that a woman has been killed -> 1/10,000
  - Many more causes of death: accidents, age, illness, husband, etc
  - Prob. of being killed given a beating husband -> 1/100
  - If we exclude other causes, the conclusion

Bayes and chance trees

- What is the probability that the first ball was black?

\[
P(W_1/W') = \frac{P(W_1)P(W'/W)}{P(W_1)P(W'/W) + P(W)P(W'/W)} = \frac{5/8 \times 3/4}{5/8 \times 3/4 + 3/8 \times 1/2} = \frac{15/16}{1/2} = \frac{3}{4}
\]

P(A/B) ≠ P(B/A)

- Speaks spanish/spanish citizenship
  \[P(\text{speaks spanish}/\text{spanish citizenship}) = \frac{40\text{million}}{40\text{million}} = 1\]
  \[P(\text{spanish citizenship}/\text{speaks spanish}) = \frac{4\text{million}}{500\text{million}} = \frac{1}{125}\]
- Kasparov/winning in chess
  \[P(\text{Kasparov winning}/\text{chess}) = \frac{1}{100\text{million}}\]
  \[P(\text{chess winning}/\text{Kasparov}) = \frac{1}{100}\]
- A more difficult one: OJ Simpson:
  - P( being killed /a beating husband)
  - P( a beating husband / being killed)

P(cancer/positive test) ≠ P(positive test/cancer)

- Problem of concept:
  - P(cancer/positive) -> Attribute of the patient
  - P(positive/cancer) -> Attribute of the test.
- What we want to know: (either)
  - Effectiveness of the treatment given how the patients fare
  - Diagnosis of the patient given the result of the test
- Diagnosis problem: P( cancer/positive test)
  - We know:
    - Clinical study: P(positive test/cancer)=0.9
    - Statistics of the population: P(cancer)=10/1000
    - Note: could be the people that go to the hospital

Game of Craps

- Dice game. The player throws two dice; if the sum is 7 or 11 wins, if it is 2, 3, or 12 loses. If it has another result, continues until a 7 (loses) or the result of the first throw (wins).

Note: the possibility of playing infinite time

Taken from: Cálculo de probabilidades I. R.Velez, V.Hernández

Game of Craps

- Justification of 28 slides for a problem:
  - Indefinite duration
  - Geometric series
  - Conditioned probability and Bayes in different ways.

Note: the possibility of playing infinite time

Game of Craps

- Compute:
  - Probability of winning
  - Probability of getting a 5 in the first throw if it is known that the player has won.
  - The probability that the player wins if there have been k throws.

Game of Craps

- Probability of winning:

\[ p = \frac{1}{36} \]
\[ q = \frac{35}{36} \]
\[ P(G_j | X = 4) = p + q^2 p + q^4 p + q^6 p + \ldots \]

Geometric Series

Games → Does not gets a 7 and repeats the first result

Game of Craps

- How do we sum an infinite geometric series?

\[ S = \frac{p}{1-q} \]
Game of Craps

• How do we sum an infinite geometric series?

\[ S = p + q + q^2 + q^3 + q^4 + q^5 + \cdots = \frac{p}{1-q} \]

Game of Craps

• How do we sum a finite geometric series?

\[ S = q + q^2 + q^3 + \cdots + q^{n-1} = \frac{q^n - q}{1-q} \]

Game of Craps

• We can sum \( P(G_1/X) = 4 \)

\[ p = P(X=4) = \frac{1}{36} \]
\[ q = P(X = (\text{NOT 7 NOR 4})) = \frac{25}{36} \]
\[ P(G_1/X = 4) = p \frac{q^4}{1-q} = \frac{1}{36} \left( \frac{25}{36} \right)^4 \]
\[ P(G_1/X = 4) = \frac{1}{36} \cdot \frac{625}{1679616} = \frac{1}{29952} \]

Note that by symmetry:

\[ P(G_1/X = 4) = P(G_1/X = 10) \]
\[ P(G_1/X = 5) = P(G_1/X = 9) \]
\[ P(G_1/X = 6) = P(G_1/X = 8) \]

Game of Craps

• The other cases

\[ p = P(X = 5) = \frac{4}{36} \]
\[ q = P(X = (\text{NOT 7 NOR 5})) = \frac{26}{36} \]
\[ P(G_1/X = 5) = p \frac{q^5}{1-q} = \frac{4}{36} \left( \frac{26}{36} \right)^5 \]
\[ P(G_1/X = 5) = \frac{4}{36} \cdot \frac{672464}{1679616} = \frac{1}{29952} \]

Note that by symmetry:

\[ P(G_1/X = 5) = P(G_1/X = 10) \]
\[ P(G_1/X = 6) = P(G_1/X = 9) \]
\[ P(G_1/X = 7) = P(G_1/X = 8) \]

Game of Craps

• Finally the probability of winning is:

\[ P(G) = P(G_1/X = 7 OR 11) + P(G_1/X = 4 OR 5 OR 6 OR 8 OR 9 OR 10) \]

\[ P(G) = \frac{2}{9} + \frac{3}{56} \cdot \frac{1}{3} + \frac{4}{5} \cdot \frac{1}{36} + \frac{5}{11} \cdot \frac{1}{36} + \frac{4}{5} \cdot \frac{1}{36} + \frac{3}{56} \cdot \frac{1}{3} \]

\[ P(G) \approx 0.493 \]

Note that by symmetry:

\[ P(G_1/X = 4) = P(G_1/X = 10) \]
\[ P(G_1/X = 5) = P(G_1/X = 9) \]
\[ P(G_1/X = 6) = P(G_1/X = 8) \]

Note that by symmetry:

\[ P(G_1/X = 7) = P(G_1/X = 8) \]
\[ P(G_1/X = 9) = P(G_1/X = 10) \]

Game of Craps

• The other cases

\[ p = P(X = 5) = \frac{4}{36} \]
\[ q = P(X = (\text{NOT 7 NOR 5})) = \frac{26}{36} \]
\[ P(G_1/X = 5) = p \frac{q^5}{1-q} = \frac{4}{36} \left( \frac{26}{36} \right)^5 \]
\[ P(G_1/X = 5) = \frac{4}{36} \cdot \frac{672464}{1679616} = \frac{1}{29952} \]

Note that by symmetry:

\[ P(G_1/X = 5) = P(G_1/X = 10) \]
\[ P(G_1/X = 6) = P(G_1/X = 9) \]
\[ P(G_1/X = 7) = P(G_1/X = 8) \]
### Game of Craps

- **Finally the probability of winning is:**
  - Note that we have used the total probability

\[
P(A) = P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \cdots + P(B_n)P(A/B_n)
\]

- **Observation:**
  - Has Won

### Game of Craps

- **Probability of getting a 5 in the first throw if it is known that the player has won.**
  - Note: inverse probability, which is the cause.

\[
P(X = 5) = \frac{P(X = 5)P(G(X = 5))}{P(G)}
\]

### Game of Craps

- **The condition is that the player wins and at least there have been \(N>k\) rounds:**
  - Note: that the temporal structure depends on getting in the first throw one element of the set={4,5,6,8,9,10}

\[
P(G|N > k) = \frac{P(N > k|X = 4) + P(N > k|X = 6) + P(N > k|X = 8) + P(N > k|X = 10)}{P(X = 4)}
\]

### Game of Craps

- **Case:**
  - Note: the condition for playing \(N>k\) rounds given that the first sum was 4 is:

\[
\begin{align*}
P(N > k / X = 4) &= \frac{P(X = 4)P(Y = \text{NOT} 7 \text{ NOR} 4)}{P(Y = \text{NOT} 7 \text{ NOR} 4)}P(\text{anything else}) \\
&= \frac{27}{36}\frac{27}{36}\frac{27}{36}\frac{27}{36} = P(N > k / X = 10) \\
&= \frac{27}{36}\frac{27}{36}\frac{27}{36}\frac{27}{36} = P(N > k / X = 9) \\
&= \frac{27}{36}\frac{27}{36}\frac{27}{36}\frac{27}{36} = P(N > k / X = 8)
\end{align*}
\]
Game of Craps

- The probability that the player wins if there have been \( k \) throws.
  - We will compute the probability of \( N > k \) which we will need for conditioning.

\[
P(N > k) = \left( P(X = 4)P(N > k | X = 4) + P(X = 5)P(N > k | X = 5) + P(X = 6)P(N > k | X = 6) \right)^2
\]

\[
P(N > k) = \left( \frac{4}{36} + \frac{5}{36} + \frac{6}{36} \right)^2
\]

Symmetry of the prob. of the set \( \{4, 5, 6, 8, 9, 10\} \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N \geq k )</td>
<td>10</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Game of Craps

- The probability that the player wins if there have been \( k \) throws.

\[
\{N > 4\} \cap G \text{ given that } \{X = 4\} \cup \{X = 10\}
\]

For \( i = k+1 \) to infinite happen either of the following events

\[
\{\text{either } 4 \text{ or } 10 \text{ happens infinite to } i\}
\]

\[
P(N > 4, X = 4) \cup \{X = 10\} = \sum_{i=2}^{\infty} \frac{6}{36} \left( \frac{27}{36} \right)^{i-1}
\]

Criterion for selecting the change of variables:

Transform the original sum into the known one

\[
S = \sum_{n=0}^{\infty} q^n = \frac{P}{1-q}
\]
Game of Craps

- The probability that the player wins *if* there have been $k$ throws.
  - Final Probability

\[
\sum_{k=3}^{10} \left[ \frac{6}{36} \binom{2}{2} + \frac{4}{36} \binom{2}{1} + \frac{10}{36} \binom{2}{0} \right] = \frac{6}{36} \binom{2}{2} + \frac{4}{36} \binom{2}{1} + \frac{10}{36} \binom{2}{0} \]

- What happens when $k \to \infty$

\[
P(G|N > k) = \lim_{k \to \infty} \frac{6}{36} \binom{2}{2} + \frac{4}{36} \binom{2}{1} + \frac{10}{36} \binom{2}{0} = \frac{1}{36} \]

Bayes and "the prosecutor's fallacy".

- The prosecutor's fallacy
  - The prosecutor's fallacy is the assertion that, because the story before the court is highly improbable, the defendant's innocence is equally improbable.
  - O.J. Simpson:
    - The chance that a random sample of DNA would match that of O.J. Simpson was put at one in 4m. Long odds: but, as Johnnie Cochran, Mr Simpson's counsel, explained to the jury, there are 20m people in the Los Angeles area. Mr Simpson was therefore one of several people whose blood might be matched to the scene and he could not be guilty beyond reasonable doubt.

\[
P(G|\text{Matches DNA}) = \frac{P(\text{Matches DNA}|G)P(G)}{P(\text{Matches DNA})}
\]

- Probability of getting a 5 in the first throw if it is known that the player has won.
  - Note: inverse probability, which is the cause.