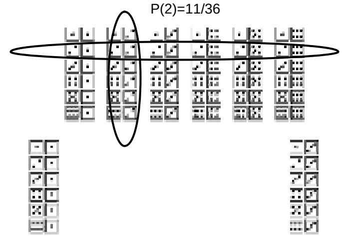
Conditioned probability

Intuition

- Time/Actions ***change*** the ***sample space***.

P(2)=11/36

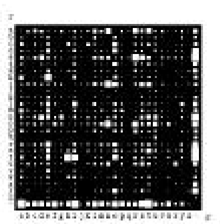


P(2/Second die is 1)=1/36

P(2/Second die is 3)=1/36

Intuition

- Time/Actions change the sample space.



http://www.inference.phy.cam.ac.uk/mackay/itila/book.htm

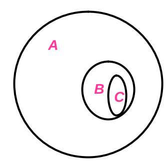
i	a_i	p_i
1	a	0.0575
2	b	0.0128
3	c	0.0263
4	d	0.0285
5	e	0.0913
6	f	0.0173
7	g	0.0133
8	h	0.0313
9	i	0.0599
10	j	0.0006
11	k	0.0084
12	l	0.0335
13	m	0.0235
14	n	0.0596
15	o	0.0689
16	p	0.0192
17	q	0.0008
18	r	0.0508
19	s	0.0567
20	t	0.0706
21	u	0.0334
22	v	0.0069
23	w	0.0119
24	x	0.0073
25	y	0.0164
26	z	0.0007
27	–	0.1928

P(k)=0.0084
P(k/y)=0
P(k/a)=0.001

Intuition

- Time/Actions change the sample space.

$$\mathrm{P}(\Omega) = Total Area = 1$$
$$\mathrm{P}(A) = Area of\ A$$
$$\mathrm{P}(C) = Area of C$$
$$\mathrm{P}(C/B) = \frac{Area of C}{Area of\ B}$$
$$\mathrm{P}(B/C) = Area of\ C$$



Formal definition

- Prob. of A conditioned to B is defined as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

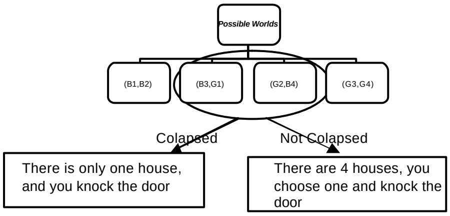
$$\Pr = \frac{Count of\ ways\ for a result}{Count of all possible results}$$

An Example: The sister problem

- You knock at the door of a family with two children, and a girl opens the door.
- Which is the probability that the other child is a boy? $\Omega = \{(B,B),(G,B),(B,G),(G,G)\}$
  - A={one child is boy}
  - B={one child is girl}

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\left(\frac{2}{4}\right)}{\left(\frac{3}{4}\right)} = \frac{2}{3}$$ Sure?

An Example: The sister problem Possible solutions



$\Omega_1 = \{\{G, B\}, \{G, G\}\}$
$P(A/B) = 1/2$

$\Omega_2 = \{(B_1, B_2), (G_1, B_3), (B_4, G_2), (G_3, G_4)\}$
$P(A/B) = 2/3$

The sister problem Possible Worlds

Logic:

Kripke In "Semantical Considerations on Modal Logic", published in 1963, Kripke responded to a difficulty with classical quantification theory. The motivation for the world-relative approach was to represent the possibility that objects in one world may fail to exist in another. If standard quantifier rules are used, however, every term must refer to something that exists in all the possible worlds. This seems incompatible with our ordinary practice of using terms to refer to things that only exist contingently.

http://en.wikipedia.org/wiki/Possible_worlds

An Example: The prisoner's dilema

(in prob. different from the game theory)

- Three prisoners A,B,C. One is going to be released, but they do not know the who.
- Prisoner A asks the guard the identity of one prisoner other than himself who **will not** be released.
- Guard:" your prob. of being released is now 1/3. If I tell you B, say, **will not** be released, then you would be one of only two prisoners whos fate is unknown and your probability of release would **increase** to ½ Since I don't want to hurt the chances of the other to be released I am not going to tell you"
  - Where is the mistake in the reasoning?

Taken from: Understanding Prob. Chance rules in everyday life. H.Tijms

An Example: The prisoner's dilema

(in prob. different from the game theory)

- Sample space $\Omega = \{\{A\}, \{B\}, \{C\}\}$
- Initial Statement: P(release A)=1/3
- The guard seems to be saying:
  - P(release A/guard says B released)=1/2
- Either the voice of the guard can change probabilities or there is a different implicit sample space or independence assumption.

Taken from: Understanding Prob. Chance rules in everyday life. H.Tijms

An Example: The prisoner's dilema

(in prob. different from the game theory)

- If we incorporate the event E={guard says B is released"}, the new sampling space is:
  - O_1={A, guard says B is released}
  - O_2={A, guard says C is released}
  - O_3={B, guard says C is released}
  - O_4={C, guard says B is released}

$O_1, O_2 \rightarrow \{A\} \rightarrow 1/3$
$O_3 \rightarrow \{B\} \rightarrow 1/3$
$O_4 \rightarrow \{C\} \rightarrow 1/3$

Taken from: Understanding Prob. Chance rules in everyday life. H.Tijms

An Example: The prisoner's dilema

(in prob. different from the game theory)

- With:
  - E_1={A is released}
  - E_2={guard says B is released}

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(O_1)}{P(O_1)+P(O_4)} = \frac{1/6}{1/6+1/3} = 1/3$$

O_1={A, guard says B is released}
O_2={A, guard says C is released}
O_3={B, guard says C is released}
O_4={C, guard says B is released}

$O_1, O_2 \rightarrow \{A\} \rightarrow 1/3$
$O_3 \rightarrow \{B\} \rightarrow 1/3$
$O_4 \rightarrow \{C\} \rightarrow 1/3$

Taken from: Understanding Prob. Chance rules in everyday life. H.Tijms

An Example: The prisoner's dilema
(in prob. different from the game theory)

- Chance tree/Decision tree

Judge decides — Guard says — Final prob.

$$1/3 \rightarrow A\ free \rightarrow 1/2 \rightarrow B\ free \rightarrow 1/6$$
$$1/3 \rightarrow A\ free \rightarrow 1/2 \rightarrow C\ free \rightarrow 1/6$$
$$1/3 \rightarrow B\ free \rightarrow 1 \rightarrow C\ free \rightarrow 1/3$$
$$1/3 \rightarrow C\ free \rightarrow 1 \rightarrow B\ free \rightarrow 1/3$$

Taken from: Understanding Prob. Chance rules in everyday life. H.Tijms

Asumptions on $\{O_1, O_2, O_3, O_4\}$

- Asumption that the original events have equal probability

Real world → Model describing the Real World →  → $\frac{P(O_1)}{P(O_1)+P(O_4)} = 1/3$

$\{O_1, O_2, O_3, O_4\}$

$O_1, O_2 \rightarrow \{A\} \rightarrow 1/3$
$O_3 \rightarrow \{B\} \rightarrow 1/3$
$O_4 \rightarrow \{C\} \rightarrow 1/3$

An Example: The prisoner's dilema
(in prob. different from the game theory)

- What would happen if:
  - E_1={A is released}
  - E_2={guard says B is released}

O_1={A, guard says B is released}
O_2={A, guard says C is released}
O_3={B, guard says B is released}
O_4={C, guard says ***B*** is released}

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(O_1)}{P(O_1)+P(O_3)+P(O_4)} = \frac{1/6}{1/6+1/3+1/3} = 1/5$$

$O_1, O_2 \rightarrow \{A\} \rightarrow 1/3$
$O_3 \rightarrow \{B\} \rightarrow 1/3$
$O_4 \rightarrow \{C\} \rightarrow 1/3$

- *Mathematical correct*
- *Independence assumption?*

Taken from: Understanding Prob. Chance rules in everyday life. H.Tijms

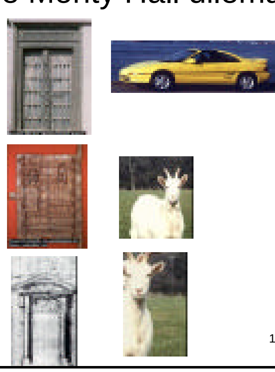
An Example: The Monty Hall dilema

- The contestant in a television show must choose between three doors. An expensive car is behind one door and gag prizes await behind the other two.
- The contestant must pick one of the doors randomly.
- The host opens one of the ther two doors concealing one of the gag prizes.
- The contestant is asked whether he wishes to switch to the remaining door.
- *Is the prob. of winning **increased** by **changing** the choosen door?*

Taken from: Understanding Prob. Chance rules in everyday life. H.Tijms

An Example: The Monty Hall dilema

Select randomly a door



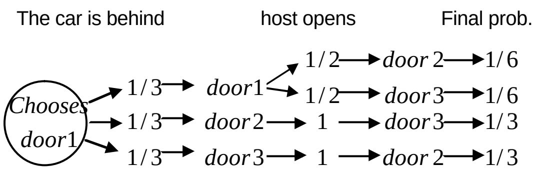
An Example: The Monty Hall dilema

- Wrong argument:
  - It makes no difference whether the player switched doors or not. Each of the two remaining unopened doors had a ½ prob. of concealing the automobile.
  - If it is not in the opened door, it is behind either, so therefore the prob. is of ½
    - ***Caveat*** is it really random?, Note that the host has choosen with knowledge.
    - Is the the same sample space?

Taken from: Understanding Prob. Chance rules in everyday life. H.Tijms

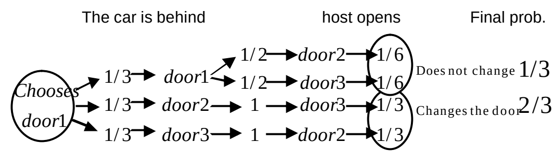
An Example: The Monty Hall dilema

- Chance tree/Decision tree



An Example: The Monty Hall dilema

- Probabilities



An Example: The Monty Hall dilema

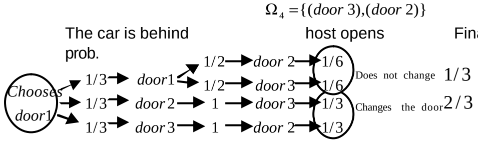
- Note that conditioning changes the sample space

$\Omega_1 = \{(\text{door }1), (door\ 3)\}$

$\Omega_{orig} = \{(\text{door}1), (door\ 2), (door\ 3)\}$ $\quad \Omega_2 = \{(\text{door}1), (door\ 2)\}$

$\Omega_3 = \{(door\ 2), (door\ 3)\}$

$\Omega_4 = \{(door\ 3), (door\ 2)\}$



An Example: The Monty Hall dilema

- Note that conditioning changes the sample space

$\Omega_{orig} = \{(\text{door}1), (door\ 2), (door\ 3)\} \longrightarrow P(\text{door }1) = 1/3$

$\Omega_1 = \{(\text{door }1), (door\ 3)\}$

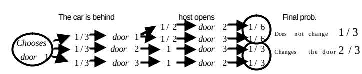
$\Omega_2 = \{(\text{door }1), (door\ 2)\}$

$\Omega_3 = \{(door\ 2), (door\ 3)\}$

$\Omega_4 = \{(door\ 3), (door\ 2)\}$

$\longrightarrow P(\text{door }1/\text{not change}) = 1/3$

$P(New\ \text{door}/\ \text{change}) = 2/3$



Computation of intersecion probability

- *Temporal structure*: B takes place and afterwards A

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(A \cap B) = P(B)P(A/B)$$

- Generalization for a sample space

$$\Omega = \{A_1, A_2, \cdots A_n\}$$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

Computation of intersecion probability

- Generalization for a sample space $\Omega = \{A_1, A_2, \cdots A_n\}$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1 \cap (A_2 \cap \cdots \cap A_n)) =$$
$$= P(A_1)P(A_2 \cap \cdots \cap A_n / A_1)$$

$$P(A \cap B) = P(A)P(B/A)$$

Computation of intersecion probability

- Generalization for a sample space

$$\Omega = \{A_1, A_2, \cdots A_n\}$$

$$P(A_1 \cap (A_2 \cap \cdots \cap A_n)) =$$
$$= P(A_1) P(A_2 \cap \cdots \cap A_n / A_1)$$

$$P(A_2 \cap (A_3 \cdots \cap A_n) / A_1) =$$
$$= P(A_2 / A_1) P(A_3 \cap \cdots \cap A_n / A_1 \cap A_2)$$

Computation of intersecion probability

- Generalization for a sample space

$$\Omega = \{A_1, A_2, \cdots A_n\}$$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) =$$
$$= P(A_1) P(A_2 / A_1) P(A_3 / A_1 \cap A_2) \cdots P(A_n / A_1 \cap \cdots \cap A_{n-1})$$

Computation of intersecion probability

- Examples
  - Birthday problem

$$\text{Sample space } \Omega = \left\{ \begin{array}{l} A_i / i = 1 ... n; \\ \text{the day of the birthday of individual i-th} \\ \textit{which} \text{ is different from all the others.} \end{array} \right\}$$

$$\text{Probability of Event} = \{A_1 \cap A_2 \cap \cdots \cap A_n\}$$
$$\downarrow$$
Probability of the simultaneous ocurrence of the atomic events $A_i \quad i = 1, \cdots n$

$P(A_2 / A_1) \rightarrow$ Probability of the birthday in day A_2 when the day A_1 is occupied

Computation of intersecion probability

- Examples
  - Birthday problem

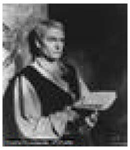
$$P(A_1 \cap A_2 \cap \cdots \cap A_n) =$$
$$= P(A_1) P(A_2 / A_1) P(A_3 / A_1 \cap A_2) \cdots P(A_n / A_1 \cap \cdots \cap A_{n-1})$$

$$\Pr = \left(\frac{N}{N}\right)\left(\frac{N-1}{N}\right)\left(\frac{N-2}{N}\right) \cdots \left(\frac{N-n+1}{N}\right)$$

Computation of intersecion probability

- Examples:Grammars for computing the probability of a correct sentence

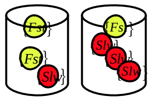
$$P(w_1 \cap w_2 \cap \cdots \cap w_n) =$$
$$= P(w_1) P(w_2 / w_1) P(w_3 / w_1 \cap w_2) \cdots P(w_n / w_1 \cap \cdots \cap w_{n-1})$$
$$= P(w_1) P(w_2 / w_1) P(w_3 / w_1 \cap w_2) P(w_4 / w_2 \cap w_3) \cdots P(w_n / w_{n-2} \cap w_{n-1})$$



To be, or not to be—that is the question;
Whether 'tis nobler in the mind to suffer
The slings and arrows of outrageous fortune,
Or to take arms against a sea of troubles,
And by opposing end them? To die, to sleep
Microsoft® Encarta®

Example

- We can connect to two servers:
  - S1 has ***2*** *high* speed links and ***1*** *slow* link
  - S2 has ***1*** *high* speed link and ***3*** *slow* links
- We select one server at random
- Which is the probability of getting a slow link?.



Example

- Sample space:

$$\Omega = \{S_1 Fst, S_1 Slw, S_2 Fst, S_2 Slw\}$$

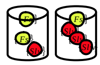
  – **Event**: Select Server and then type of link

Chooses Server

$1/2 \rightarrow S_1$: $2/3 \rightarrow HghSpeed$; $1/3 \rightarrow LowSpeed \leftarrow$

$1/2 \rightarrow S_2$: $1/4 \rightarrow HghSpeed$; $3/4 \rightarrow LowSpeed \leftarrow$

Prob=1/2*1/3+1/2*3/4=13/24

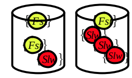


Example

- Probabilities:

$$P(S_1) = P(S_2) = 1/2$$

$$P(Slw/S_1) = \frac{1}{3} = \frac{P(Slw \cap S_1)}{P(S_1)}$$



$$P(Slw/S_2) = \frac{3}{4} = \frac{P(Slw \cap S_2)}{P(S_2)}$$

$$P(Slw) = P(S_2)P(Slw/S_2) + P(S_1)P(Slw/S_1)$$

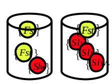
Example

- Probabilities: Another way

$$P(Slw) = P\big((Slw \cap S_1) \cup (Slw \cap S_2)\big)$$

$$P(Slw) = P(Slw \cap S_1) + P(Slw \cap S_2)$$

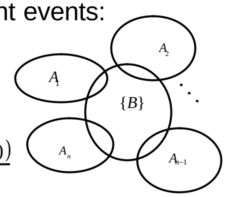
$$P(Slw) = P(S_2)P(Slw/S_2) + P(S_1)P(Slw/S_1)$$



Properties

- If we have a set of disjoint events:

$$S = \{A_1, A_2, \cdots A_n\}$$



$$P(A_1 \cup A_2 \cup \cdots \cup A_n / B) =$$

$$= \frac{P\big((A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)\big)}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)}{P(B)}$$

Properties

- The universe conditioned to a given event has prob. one:

$$P(\Omega / B) = \frac{P(\Omega \cap B)}{P(B)} = 1$$

- Both prob. spaces will have the same properties

$$(\Omega, P(./B)) \qquad (B, P(./B))$$

Properties

- Total probabilities:
  – Given a set of events $S = \{B_1, B_2, \cdots B_n\}$ pairwise disjoint or mutually disjoint[1] , such that
  – We have: $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$

$$P(A) = P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \cdots + P(B_n)P(A/B_n)$$

1 - MutuallyDisjoint: for all (i, j) such that $i \neq j$

$B_i \cap B_j = 0$

Properties

- Total probabilities:
  - Proof:

$$A = A\cap\Omega = A\cap(B_1\cup B_2\cup\cdots\cup B_n) =$$
$$= (A\cap B_1)\cup(A\cap B_2)\cup\cdots\cup(A\cap B_n)$$
$$P(A) = P(A\cap B_1)\cup P(A\cap B_2)\cup\cdots\cup P(A\cap B_n)$$
$$P(A) = P(A\cap B_1) + P(A\cap B_2) + \cdots + P(A\cap B_n)$$
$$but\quad P(A\cap B_i) = P(B_i)P(A/B_i)$$
$$P(A) = P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \cdots + P(B_n)P(A/B_n)$$

Properties

- Total probabilities:
  - Temporal/Spacial interpretation

$$P(A) = P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \cdots + P(B_n)P(A/B_n)$$

  - When B_i happens, the odds of A change

$$A \begin{array}{l} \nearrow P(B_1) \longrightarrow P(A/B_1) \\ \rightarrow P(B_2) \longrightarrow P(A/B_2) \\ \quad\ \vdots \qquad\qquad\quad \vdots \\ \searrow P(B_n) \longrightarrow P(A/B_n) \end{array} \Bigg\} P(A)$$

Example

- A Clinical analysis is used for the diagnosis of three illness, B_1,B_2,B_3.
- The proportion of people with a given illness is: 3%,2%,1%
- The analysis gives possitive result for:

$$\left.\begin{array}{ll} B_1 \rightarrow & 85\% \\ B_2 \rightarrow & 92\% \\ B_3 \rightarrow & 78\% \\ B_0 \rightarrow & 0.5\% \end{array}\right\}$$

- Compute the prob. of a possitive.

Example

- We define prob. of a possitive as P(A)

$$P(A) = P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + P(B_3)P(A/B_3) + P(B_0)P(A/B_0)$$
$$= 0.03*0.85+0.02*0.92+0.01*0.78+0.94*0.005=0.0564$$

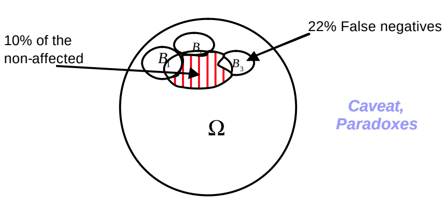
- ***5,6% instead of 6% not bad*** ☺
- What would happen with $P(A/B_0)=0.1$?

Example

- What would happen with $P(A/B_0)=0.1$?

$$P(A) = P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + P(B_3)P(A/B_3) + P(B_0)P(A/B_0)$$
$$= 0.03*0.85+0.02*0.92+0.01*0.78+0.94*0.1=0.146$$

- Why 14.6%? ***Only 6% should give possitive!!!!!***



Bayes's formula

- Intuitively not clear (?)

$$P(B_i/A) = \frac{P(B_i)P(A/B_i)}{P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \cdots + P(B_n)P(A/B_n)}$$

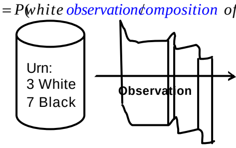
- Context:
  - Solution to a problem of "inverse probability" was presented in the *Essay Towards Solving a Problem in the Doctrine of Chances*
  - Published *Divine Benevolence, or an Attempt to Prove That the Principal **End** of the Divine Providence and Government is the Happiness of His Creatures* (?)
  - **End** is synomim of **purpose, aim**.

Purpose of the Divine Providence $\longrightarrow$ Observation: Happines of His Creatures

Kinds of probability

- Probability of an observation
- ***Probability of the cause of the observation***
- Probability of the estimate of the probability.

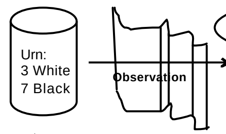
$p_{\text{Real}}^* = P(\textit{white observation}/\textit{composition of the urn})$



$P_{New}(\text{composition of the urn}/\text{ white observation})$

Bayes Formula

- Motivation:
  - We would like the probability of the causes that generate the observations.
  - Update our knowledge after the observations.



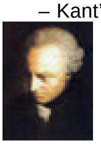
$P(\text{Urn with } n \text{ white and } m \text{ black})$
?

$P_{New}(\textit{composition of the urn} / \text{Given the observation})$

$p_{\text{Real}}^* = P(A \text{ given observation} / \textit{composition of the urn})$

Bayes Formula

- Context of the Bayes formula:
  - Kant's knowledge theory

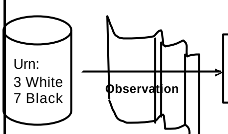


http://en.wikipedia.org/wiki/Kant

$P(\text{Urn with } n \text{ white and } m \text{ black})$

Senses are composed of:

A priori forms, are Believes on the composition (i.e. n white and m black)



Transformation of the a priori knowledge by the observation

$P(\textit{composition of the urn} /$ Given the observation)

Gives form to the knowledge

$p^* = P(\textit{white observation}/\textit{composition of the urn})$

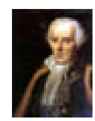
Bayes Formula

- Laplace proposes taking into account the probability of the probability:

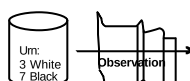
$P(\text{Urn with } n \text{ white and } m \text{ black})$

Senses are composed of:

A priori forms,



http://en.wikipedia.org/wiki/Laplace



Transformation of the a priori knowledge by the observation

$P_{New}(\textit{Observation}) = \textit{function}(\text{Apriori knowledge, Observation})$

$p^* = P(\textit{white observation}/\textit{composition of the urn})$

Bayes' formula

- Bayes' formula:
  - Given a set of events $S = \{B_1, B_2, \cdots B_n\}$ pairwise disjoint or mutually disjoint[1] , such that $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$
  - We have: $P(B_i / A) = \dfrac{P(B_i \cap A)}{P(A)} = \dfrac{P(B_i)P(A/B_i)}{P(A)}$

$$P(B_i / A) = \frac{P(B_i)P(A/B_i)}{P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \cdots + P(B_n)P(A/B_n)}$$

1 - Mutually Disjoint : for all (i, j) such that $i \neq j$
$B_i \cap B_j = 0$

Bayes's formula

- Context:
  - Bayes defines *probability* as follows :
    - *The probability of any event is the ratio between the* ***value at which an expectation*** *depending on the happening of the event ought to be computed, and the* ***chance of the thing expected*** *upon it's happening*

Expectation depending on the happening

Chance of the thing expected upon it's happening

$$P(B_i / A) = \frac{P(B_i)\ \ P(A/B_i)}{P(B_1)P(A/B_1) + P(B_2)P(A/B_2) + \cdots + P(B_n)P(A/B_n)}$$

Bayes's formula

- Expectation
  - **1. anticipation of something happening:** a confident belief or strong hope that a particular event will happen
  - **2. notion of something:** a mental image of something expected, often compared to its reality (*often used in the plural*)
  - **3. expected standard:** a standard of conduct or performance expected by or of somebody (*often used in the plural*)
  -

Bayes's formula and *Utility function*

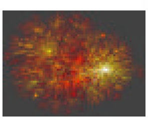
- Idea of utility
  - Expectation is a subjective concept.
  - Instead of the probabilistic expectation, we could think in the ***'moral expectation'***

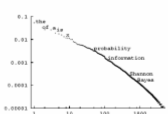
Value $\rightarrow$ Chance of the thing X Moral expectation depending on the happening

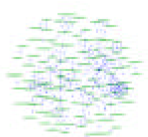
$$Value(B_i / A) \propto P(B_i)\ Utility(A / B_i)$$

Bayes's formula and *Utility function*

- Usefullness of the Idea of utility
  - People behave as if they were maximizing an Utility function, i.e, a ***'moral expectation'***
  - Objects generated by the independent and free actions can be understood by means of the utility theory.
  - Example: $Value(B_i / A) \propto P(B_i)\ Utility(A / B_i)$







Application to the clinical analysis problem

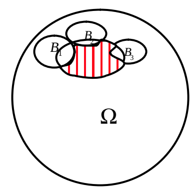
- Probability of having the illness given a possitive result (***A***).

$$P(B_1 / A) = \frac{P(B_1)P(A/B_1)}{P(A)} = \frac{0.03 * 0.85}{0.0564} = 0.452$$

$$P(B_2 / A) = \frac{P(B_2)P(A/B_2)}{P(A)} = \frac{0.02 * 0.92}{0.0564} = 0.326$$

$$P(B_3 / A) = \frac{P(B_3)P(A/B_3)}{P(A)} = \frac{0.01 * 0.78}{0.0564} = 0.138$$

$$P(B_0 / A) = \frac{P(B_0)P(A/B_0)}{P(A)} = \frac{0.94 * 0.05}{0.0564} = 0.083$$



Application to the clinical analysis problem

- Probability of having the illness given a possitive result (***A***).
- Meaning:

$$0.03 \quad 0.85 \quad 0.0564$$

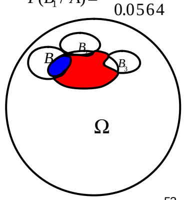
$$\frac{P(B_1 \cap A)}{P(A)} \quad P(B_1 / A) = \frac{0.03 * 0.85}{0.0564}$$

$P(B_1 / A) \rightarrow 45\%$

$P(B_2 / A) \rightarrow 32\%$

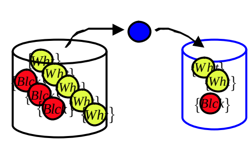
$P(B_3 / A) \rightarrow 13\%$

$P(B_0 / A) \rightarrow 8.3\%$ False positive



Bayes and chance trees

- Urn has 5 white balls and 3 black. One ball is taken randomly and introduced in another with 2 white and 1 black.
- A ball **white** is taken from the second urn.
- What is the probability that the first ball was black?.



Bayes and chance trees

- Urn has 5 white balls and 3 black. One ball is taken randomly and introduced in another with 2 white and 1 black.
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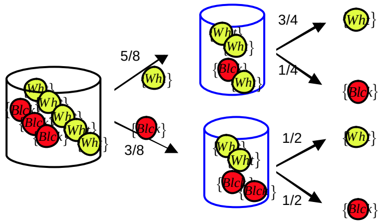
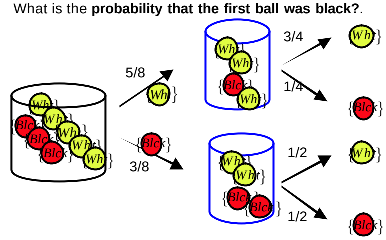


Bayes and chance trees

- What is the **probability that the first ball was black?**.



$$P(B_1/W_2) = \frac{P(B_1)P(W_2/B_1)}{P(B_1)P(W_2/B_1)+P(W_1)P(W_2/W_1)} = \frac{3/8*1/2}{3/8*1/2+5/8*3/4}$$

P(A/B) ≠ P(B/A)

- Speaks spanish/spanish citzenship

$$P(\text{speaks spanish}/\text{spanish citzenship}) = \frac{40 million}{40 million} = 1$$

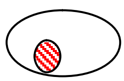
$$P(\text{spanish citzenship}/\text{speaks spanish}) = \frac{40 million}{500 million} \cong \frac{4}{5}$$

- Kasparov/winning in chess

$$P(\text{Kasparov}/\text{winning in chess}) = \frac{1}{1000 million} \cong 0$$

$$P(\text{winning in chess}/\text{Kasparov}) = 1$$

- A more difficult one: OJ Simpson:
  - P(being killed /a beating husband)
  - P(a beating husband / being killed)



Bayes and "the prosecutor's fallacy".

•The prosecutor's fallacy

•The prosecutor's fallacy is the assertion that, because the story before the court is highly improbable, the defendant's innocence is equally improbable.

•OJ Simpson:

•Prob. Having a beating husband **given** that a woman has been killed->1/10.000

•Many more causes of death: accidents,age,illness, husband,etc

•Prob. of being killed **given** a beating husband->1/100

•If we exclude other causes, the conclusion

P(cancer/positive test) ≠ P(positive test/cancer)

- Problem of concept:
  - P(cancer/positive) ->Atribute of the pacient
  - P(positive/cancer) ->Atribute of the test.
- What we want to know: (either)
  - **Efectivness** of the treatment given how the pacients fare
  - **Diagnosis** of the pacient given the result of the test
- **Diagnosis** problem: P(**cancer**/positive test)
  - We know:
  - Clinical study: P(positive test/cancer)=0.9
  - Statistics of the population: P(cancer)=10/1000
    - Note: could be the people that go the hospital

Taken from: Understanding Prob. Chance rules in everyday life. H.Tijms

P(cancer/positive test) ≠ P(positive test/cancer)

- **Diagnosis** problem: P(**cancer**/positive)
  - We know:
    - Clinical study: P(positive test/cancer)=0.9
    - Statistics of the population: P(cancer)=10/1000

$$\text{We define} \rightarrow \begin{cases} C: \text{Pacient is ill of cancer} \\ P: \text{Positive result in the test} \end{cases}$$

$$P(C/P) = \frac{P(C)P(P/C)}{P(C)P(P/C)+P(\bar{C})P(P/\bar{C})} = \frac{\frac{10}{1000}0.90}{\frac{10}{1000}0.90+\frac{990}{1000}0.10} = 0.0833$$

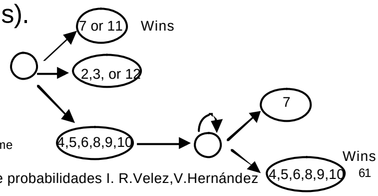
Note:

•90% vs. 8.3%

•1% is increased to 8,3%

Game of Craps

- Dice game. The player throws two dice; if the sum is 7 or 11 wins, if it is 2,3, or 12 loses. If it has another result, continues until a 7 (loses) or the result of the first throw (wins).

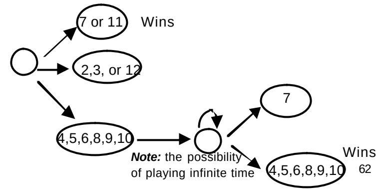


Note: the possibility of playing infinite time

Taken from: Cálculo de probabilidades I. R.Velez,V.Hernández

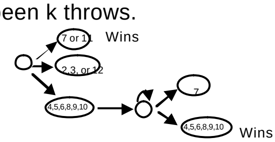
Game of Craps

- Justification of 28 slides for a problem:
  - Indefinite duration
  - Geometric series
  - Conditioned probability and Bayes in different ways.



Game of Craps

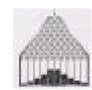
- Compute:
  - Probability of winning
  - Probability of getting a 5 in the first throw if it is known that the player has won.
    - Note: inverse probability, which is the cause.
  - The probability that the player wins if there have been k throws.

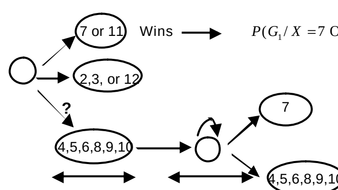


Game of Craps

- Probability of winning :

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36





$P(G_1 / X = 7 \text{ OR } X = 11) = \frac{6}{36} + \frac{2}{36} = \frac{2}{9}$

Wins-> Does not gets a 7 and repeats the first result

$P(G_1)$ $P(G_2)$

Game of Craps

- Probability of winning :

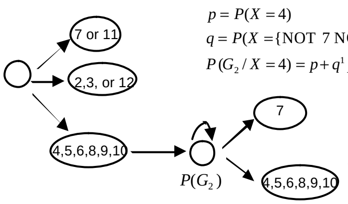
2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$p = P(X = 4)$

$q = P(X = \{\text{NOT } 7 \text{ NOR } 4\})$

$P(G_2 / X = 4) = p + q^1 p + q^2 p + q^3 p + q^4 p \cdots$

Geometric Series

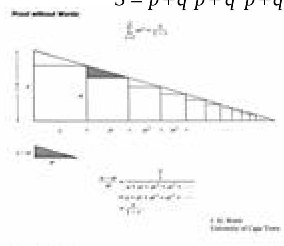


$P(G_2)$

Wins-> Does not gets a 7 and repeats the first result

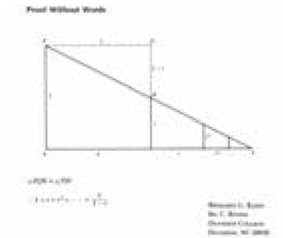
Game of Craps

- How do we sum an infinite geometric series? $S = p + q^1 p + q^2 p + q^3 p + q^4 p \cdots = \frac{p}{1-q}$



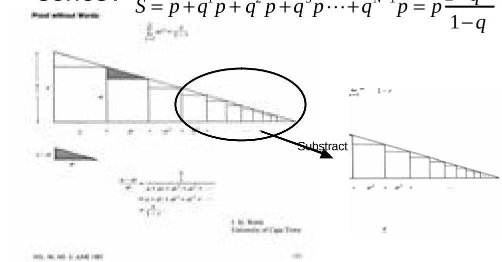
Game of Craps

- How do we sum an infinite geometric series? $S = p + q^1 p + q^2 p + q^3 p + q^4 p \cdots = \frac{p}{1-q}$



Game of Craps

- How do we sum an ***finite*** geometric series? $S = p + q^1 p + q^2 p + q^3 p \cdots + q^{N-1} p = p\frac{1-q^N}{1-q}$



Game of Craps

- How do we sum an ***finite*** geometric series?
  - Analitical Solution

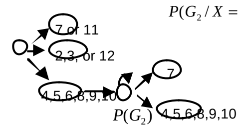
$$S = 1 + q^1 + q^2 + q^3 \cdots + q^{N-1}$$
$$S = 1 + q^1 + q^2 + q^3 \cdots + q^{N-1} + \left(q^N - q^N\right)$$
$$S = 1 + q^1\left(1 + q^1 + q^2 + q^3 \cdots + q^{N-1}\right) - q^N$$
$$S = 1 + qS - q^N$$
$$S = \frac{1-q^N}{1-q}$$

Game of Craps

- We can sum $P(G_2 / X = 4)$

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$$p = P(X = 4) = \frac{3}{36}$$
$$q = P(X = \{\text{NOT } 7 \text{ NOR } 4\}) = \frac{27}{36}$$
$$P(G_2 / X = 4) = p\frac{1-q^N}{1-q} = \frac{3}{36}\frac{1}{1-\frac{27}{36}} = \frac{1}{3}$$



Wins-> Does not gets a 7 and repeats the first result

Game of Craps

- The other cases

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$$p = P(X = 5) = \frac{4}{36}$$
$$q = P(X = \{\text{NOT } 7 \text{ NOR } 5\}) = \frac{26}{36}$$
$$P(G_2 / X = 5) = p\frac{1}{1-q} = \frac{4}{36}\frac{1}{1-\frac{26}{36}} = \frac{2}{5}$$

$$p = P(X = 6) = \frac{5}{36}$$
$$q = P(X = \{\text{NOT } 7 \text{ NOR } 6\}) = \frac{25}{36}$$
$$P(G_2 / X = 6) = p\frac{1}{1-q} = \frac{5}{36}\frac{1}{1-\frac{25}{36}} = \frac{5}{11}$$

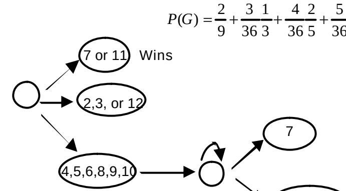
Note that by symetry:

$$P(G_2 / X = 4) = P(G_2 / X = 10)$$
$$P(G_2 / X = 5) = P(G_2 / X = 9)$$
$$P(G_2 / X = 6) = P(G_2 / X = 8)$$

Game of Craps

- Finally the probability of winning is:

$$P(G) = P(G_1 / X = 7 \text{ OR } 11) + P(G_1 / X = 4 \text{ OR } 5 \text{ OR } 6 \text{ OR } 8 \text{ OR } 9 \text{ OR } 10)$$
$$P(G) = \frac{2}{9} + \frac{3}{36}\frac{1}{3} + \frac{4}{36}\frac{2}{5} + \frac{5}{36}\frac{5}{11} + \frac{4}{36}\frac{2}{5} + \frac{3}{36}\frac{1}{3} \cong 0.493$$



Wins-> Does not gets a 7 and repeats the first result

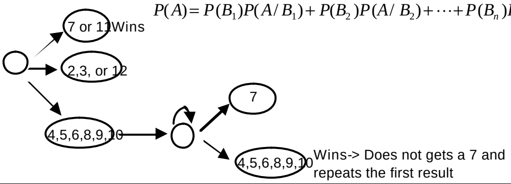
2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Game of Craps

- Finally the probability of winning is:
  – Note that we have used the total probability

$$P(G)=\frac{2}{9}+\underbrace{\frac{3}{36}\frac{1}{3}+\frac{4}{36}\frac{2}{5}+\frac{5}{36}\frac{5}{11}+\frac{4}{36}\frac{2}{5}+\frac{3}{36}\frac{1}{3}}_{Total\ Probability}\cong 0.493$$

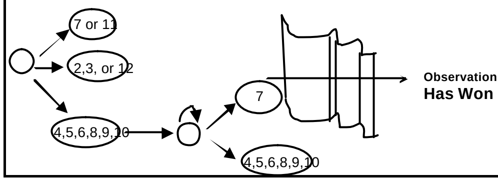
$$P(A)=P(B_1)P(A/B_1)+P(B_2)P(A/B_2)+\cdots+P(B_n)P(A/B_n)$$



Game of Craps

– Probability of getting a 5 in the first throw ***if*** it is ***known*** that the player has ***won***.
  - Note: inverse probability, which is the cause?.

$$P(\{X=5\}/G)=\frac{P(\{X=5\})P(G/\{X=5\})}{P(G)}=\frac{\frac{4}{36}\frac{2}{5}}{0.493}=\frac{11}{122}$$



Game of Craps

- The probability that the player wins if there have been ***k*** throws.
- The condition is that the player wins and at least there have been N>k rounds: $P(G/\{N>k\})$
  – Note: that the temporal structure depends on getting in the first throw one element of the set={4,5,6,8,9,10}

$$P(G/\{N>k\}) \longrightarrow \begin{cases} P(N>k/X=4) \\ P(N>k/X=5) \\ P(N>k/X=6) \\ P(N>k/X=8) \\ P(N>k/X=9) \\ P(N>k/X=10) \end{cases}$$

Game of Craps

- The probability that the player wins if there have been ***k*** throws.
- Case: $P(N>k/X=4)$
  – Note: the condition for playing N>k rounds given that the first sum was 4 is:

$$\{Get\ a\ 4\}\ AND\ \underbrace{\{\text{NOT 7 NOR 4}\}\ AND\cdots AND\ \{\text{NOT 7 NOR 4}\}}_{k-1\ \text{times}}\ AND\ \{\text{after anything else}\}$$

$$P(N>k/X=4)=\frac{P(X=4)P(X=\{\text{NOT 7 NOR4}\})^{k-1}}{P(X=4)}P(\text{anything else})$$

Game of Craps

- The probability that the player wins if there have been ***k*** throws.

$$P(N>k/X=4)=\frac{P(X=4)P(X=\{\text{NOT 7 NOR4}\})^{k-1}}{P(X=4)}P(\text{anything else})$$

$$P(X=\{\text{NOT 7 NOR4}\})=\frac{27}{36}$$

$$P(N>k/X=4)=(\frac{27}{36})^{k-1}=P(N>k/X=10)$$

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Game of Craps

- The probability that the player wins if there have been ***k*** throws.
  – For the rest of the set={4,5,6,8,9,10}

$$P(N>k/X=4)=(\frac{27}{36})^{k-1}=P(N>k/X=10)$$

$$P(N>k/X=5)=(\frac{26}{36})^{k-1}=P(N>k/X=9)$$

$$P(N>k/X=6)=(\frac{25}{36})^{k-1}=P(N>k/X=8)$$

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Game of Craps

- The probability that the player wins ***if*** there ***have been k*** throws.
  - We will compute the probaby of N>k which we will need for ***conditioning***.

$$P(N>k)=\left(P(X=4)P(N>k/X=4)+P(X=5)P(N>k/X=5)+P(X=6)P(N>k/X=6)\right)*2$$

$$P(N>k)=\frac{6}{36}\left(\frac{27}{36}\right)^{k-1}+\frac{8}{36}\left(\frac{26}{36}\right)^{k-1}+\frac{10}{36}\left(\frac{25}{36}\right)^{k-1}$$

Symetry of the prob. of the set={4,5,6,8,9,10}

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Game of Craps

- The probability that the player wins ***if*** there ***have been k*** throws.

$$\{N>k\}\cap G \text{ given that} \{X=4\}\cup\{X=10\}$$

For $i=k+1$ to infinite happens either of the following events

$$\{Get\ a\ 4\ OR\ 10\}\ AND\ \underbrace{\{NOT\ 7\ NOR\ (4,10)\}\ AND \cdots AND\ \{NOT\ 7\ NOR\ (4,10)\}}_{i-2 \text{ times}}\ AND\ \{win\}$$

$$i \geq k+1$$

$$P(\{N>k\}\cap G/\{X=4\}\cup\{X=10\})=\sum_{i=k+1}^{\infty}\frac{6}{36}\left(\frac{27}{36}\right)^{i-2}$$

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- The probability that the player wins ***if*** there ***have been k*** throws.

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$$\{Get\ a\ 4\ OR\ 10\}\ AND\ \underbrace{\{NOT\ 7\ NOR\ (4,10)\}\ AND \cdots AND\ \{NOT\ 7\ NOR\ (4,10)\}}_{i-2 \text{ times}}\ AND\ \{win\}$$

$$i \geq k+1$$

$$P(\{N>k\}\cap G/\{X=4\}\cup\{X=10\})=\sum_{i=k+1}^{\infty}\frac{6}{36}\left(\frac{27}{36}\right)^{i-2}$$

$$\frac{3}{36}+\frac{3}{36}=\frac{6}{36}$$

Geometric Series

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Game of Craps

- Sum of the geometric series

$$\frac{6}{36}\sum_{i=k+1}^{\infty}\left(\frac{27}{36}\right)^{i-2}\underset{l\to i-(k+1)}{\overset{c.v.}{=}}\frac{6}{36}\sum_{l=0}^{\infty}\left(\frac{27}{36}\right)^{l+k-1}=\frac{6}{36}\left(\frac{27}{36}\right)^{k-1}\sum_{l=0}^{\infty}\left(\frac{27}{36}\right)^{l}$$

$$=\frac{6}{36}\left(\frac{27}{36}\right)^{k-1}\frac{1}{1-\frac{27}{36}}=\frac{6}{36}\left(\frac{27}{36}\right)^{k-1}\frac{36}{36-27}=\frac{1}{3}\left(\frac{27}{36}\right)^{k-1}$$

Criterion for selecting the change of variables: Transform the original sum into the known one

$$S=p\sum_{n=0}^{\infty}q^{n}=\frac{p}{1-q}$$

Game of Craps

- The probability that the player wins ***if*** there ***have been k*** throws.

$$P(\{N>k\}\cap G/\{X=4\}\cup\{X=10\})=\sum_{i=k+1}^{\infty}\frac{3}{36}\left(\frac{27}{36}\right)^{i-2}=\frac{1}{3}\left(\frac{27}{36}\right)^{k-1}$$

$$P(\{N>k\}\cap G/\{X=5\}\cup\{X=9\})=\sum_{i=k+1}^{\infty}\frac{4}{36}\left(\frac{26}{36}\right)^{i-2}=\frac{4}{10}\left(\frac{26}{36}\right)^{k-1}$$

$$P(\{N>k\}\cap G/\{X=6\}\cup\{X=8\})=\sum_{i=k+1}^{\infty}\frac{5}{36}\left(\frac{25}{36}\right)^{i-2}=\frac{5}{11}\left(\frac{25}{36}\right)^{k-1}$$

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Game of Craps

- The probability that the player wins ***if*** there ***have been k*** throws.

$$P(\{N>k\}\cap G)=\frac{6}{36}\frac{1}{3}\left(\frac{27}{36}\right)^{k-1}+\frac{8}{36}\frac{4}{10}\left(\frac{26}{36}\right)^{k-1}+\frac{10}{36}\frac{5}{11}\left(\frac{25}{36}\right)^{k-1}$$

Total probability

$$P(A)=P(B_1)P(A/B_1)+P(B_2)P(A/B_2)+P(B_3)P(A/B_3)$$

$$P(\{N>k\}\cap G/\{X=4\}\cup\{X=10\})$$

$$P(\{N>k\}\cap G/\{X=5\}\cup\{X=9\})$$

$$P(\{N>k\}\cap G/\{X=6\}\cup\{X=8\})$$

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Game of Craps

- The probability that the player wins ***if*** there ***have been k*** throws.
  - Final Probability

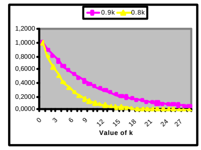
$$P(G/\{N>k\}) = \frac{P(\{N>k\}\cap G)}{P(N>k)} =$$

$$= \frac{\frac{6}{36}\frac{1}{3}\left(\frac{27}{36}\right)^{k-1} + \frac{8}{36}\frac{4}{10}\left(\frac{26}{36}\right)^{k-1} + \frac{10}{36}\frac{5}{11}\left(\frac{25}{36}\right)^{k-1}}{\frac{6}{36}\left(\frac{27}{36}\right)^{k-1} + \frac{8}{36}\left(\frac{26}{36}\right)^{k-1} + \frac{10}{36}\left(\frac{25}{36}\right)^{k-1}}$$

Game of Craps

- The probability that the player wins ***if*** there ***have been k*** throws.
  - What happens when k->infinite

Value of k	*0.9^k*	*0.8^k*
0	1,0000	1,0000
1	0,9000	0,8000
2	0,8100	0,6400
3	0,7290	0,5120
4	0,6561	0,4096
5	0,5905	0,3277
6	0,5314	0,2621
7	0,4783	0,2097
8	0,4305	0,1678
9	0,3874	0,1342
10	0,3487	0,1074
11	0,3138	0,0859



$$P(G/\{N>k\}) \cong \frac{\frac{6}{36}\frac{1}{3}\left(\frac{27}{36}\right)^{k-1} + e}{\frac{6}{36}\left(\frac{27}{36}\right)^{k-1} + d} \cong \frac{1}{3}$$

Bayes and "the prosecutor's fallacy".

- The prosecutor's fallacy
  - The prosecutor's fallacy is the assertion that, because the story before the court is highly improbable, the defendant's innocence is equally improbable.
- OJ Simpson:
  - The chance that a random sample of DNA would match that of O.J. Simpson was put at one in 4m. Long odds: but, as Johnnie Cochran, Mr Simpson's counsel, explained to the jury, there are 20m people in the Los Angeles area. Mr Simpson was therefore one of several people whose blood might be matched to the scene and he could not be guilty beyond reasonable doubt.

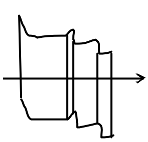
$$P(\text{Guilty/ DNA Matches}) = \frac{P(\text{Guilty})P(\text{DNA Matches/ Guilty})}{P(\text{Guilty})P(\text{DNA Matches/ Guilty}) + P(\textit{Not}\ \text{Guilty})P(\text{DNA Matches/ }\textit{Not}\ \text{Guilty})}$$

$$P(\text{DNA Matches/Guilty}) = \frac{P(\text{DNA Matches})P(\text{DNA Matches / Guilty})}{P(\text{DNA Matches})P(\text{Guilty / DNA don't Match}) + P(\text{DNA don't Match})P(\text{Guilty/DNA dont Match})}$$

REVISSAR!!!!!

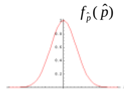
What is the chance of your being guilty?
Financial Times 19 June 2003

- Probability of getting a 5 in the first throw if it is known that the player has won.
  - Note: inverse probability, which is the cause.

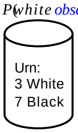


Observation:
Has Won

$f_{\hat{p}}(\hat{p})$



$p_{Real}^{*} = P(white\ observation/composition\ of\ the\ urn)$



p^{*}

$P_{New}(composition\ of\ the\ urn/\ white\ observation)$