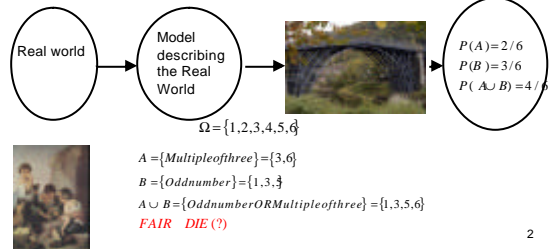


# Allocation of probabilities



# Allocation of probabilities

- Construct a bridge between reality and the mathematical model of probability



# Laplace's Rule

- Premise on Laplace's Rule:
- Principle of insufficient reason
  - If we do not know the probabilities, we should suppose that all possibilities are equally likely.
  - Problem: the probabilities depend on how a sample space is **defined**.

$$\Omega = \{She - Cat, He - Cat\} \rightarrow \{1/2, 1/2\}$$

$$\Omega' = \{She - Cat, He - Cat, I - don't - know\} \rightarrow \{1/3, 1/3, 1/3\}$$

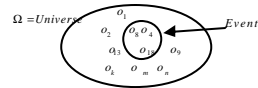
- Important in *Decision Theory*. Savage Principles of Statistics

# Laplace's Rule

- Laplace's Rule:

$$Pr = \frac{\text{Count of ways for a result}}{\text{Count of all possible results}}$$

- If **all** possible cases have the **same** probability.

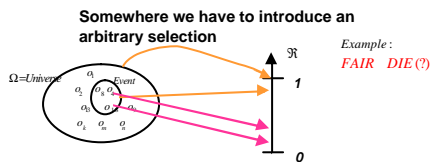


# Laplace's Rule

- Problem of *self-reference*:

$$Pr = \frac{\text{Count of ways for a result}}{\text{Count of all possible results}}$$

- If **all** possible cases have the **same** probability.



# Laplace's Rule

- If we know that an outcome is more possible than another *i.e.*  $o_2$  we create new labels for the outcome.

$$\Omega = \{o_1, o_2, \dots, o_n\} \rightarrow \Omega' = \{o_1, o_2^1, o_2^2, \dots, o_2^m, \dots, o_n\}$$

$$Pr = \frac{\text{Count of ways for a result}}{\text{Count of all possible results}}$$

## Example 1

- A bag with 6 balls.
  - 3 Red, 2 Green, 1 white.
  - Compute the prob. of extracting each colour.

$$\Omega = \{R_1, R_2, R_3, V_1, V_2, W_1\}$$

Events

$$A = \{R_1, R_2, R_3\}$$

$$B = \{V_1, V_2\}$$

$$C = \{W_1\}$$

$$\Pr(A) = \frac{3}{6}$$

$$\Pr(B) = \frac{2}{6}$$

$$\Pr(C) = \frac{1}{6}$$

Example taken from VELEZ, HERNÁNDEZ, Cálculo de Probabilidades

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## Example 2

- An Urn contains three cards, one with both sides red, one with both sides white, and a third with one side red and the other white.
- We extract a card, and examine **only one** side.
- Compute the probability that the **unseen side** is of the **same** colour.
  - Mistaken line of reasoning (D'Alembert's mistake)
  - The unseen **outcome** is that "side can be of equal or different to the observed"
  - This implies that both results have the same probability, and that the sample space is  $\Omega = \{equal, different\}$

Example taken from VELEZ, HERNÁNDEZ, Cálculo de Probabilidades

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## Example 2

- Real sample space. Take into account that the outcome is to set a given face upwards.

– Name each face

$$\Omega = \{R_1, R_2, R_3, R_4, R_5, R_6, W_1, W_2, W_3, W_4, W_5, W_6\}$$

- If each outcome is equally likely  $\rightarrow \Pr = 1/6$
- Event = {two faces with the same colour}

$$A = \{R_1, R_2, R_3, R_4, R_5, R_6, W_1, W_2, W_3, W_4, W_5, W_6\} \quad \Pr(A) = \frac{4}{6}$$

$$B = \{R_3, W_3, R_3\} \quad \Pr(B) = \frac{2}{6}$$

Two out of three will be equal<sup>9</sup>

Example taken from VELEZ, HERNÁNDEZ, Cálculo de Probabilidades

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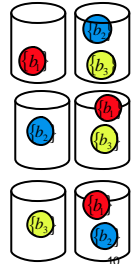
## Example 3

- Three balls are put in two urns  $\{I, D\}$ .
- Probability that  $I$  has at least a ball?

– Mistake:

- Possible sample space  $\Omega = \{0, 1, 2, 3\}$
- Note: there is only one way of introducing 3 balls in  $\{I\}$

– if there is only one, it could be the first, second or third.



Example taken from VELEZ, HERNÁNDEZ, Cálculo de Probabilidades

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## Example 3

- Distinguish the ball by the introduction moment

$$\{b_1, b_2, b_3\}$$

– Sample space

$$\Omega = \{III, IID, IDI, IDI, DDI, DDI, DID, DID, DDI, DDD\}$$

– Now each outcome has the probability  $\rightarrow 1/8$

– Event  $A = \{\text{At least one ball in } I\}$

$$A = \{III, IID, IDI, IDI, DDI, DDI, DID, DID\}$$

$$\Pr(A) = \frac{7}{8}$$

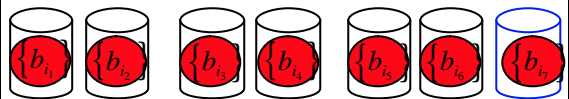


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## Combinatorial Methods. Lotto6/49

- Lotto6/49: 6 numbers+ 1 complementary are selected from 49.
  - Probability of guessing k
  - Probability of guessing k AND the complementary
  - Probability of guessing k AND Not the complementary

$$\{b_1, b_2, b_3, \dots, b_{48}, b_{49}\} \rightarrow \{i_1, i_2, \dots, i_7\} \rightarrow \{b_{i_1}, b_{i_2}, \dots, b_{i_7}\}$$

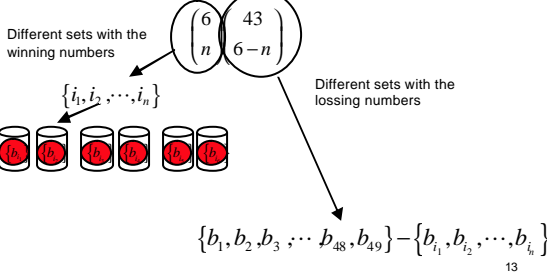


Example taken from VELEZ, HERNÁNDEZ, Cálculo de Probabilidades

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## Combinatorial Methods.Lotto6/49

- Number of ways for guessing n results.



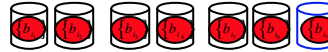
## Combinatorial Methods.Lotto6/49

- Probability of guessing k

$$\Pr(n) = \frac{\binom{43}{6-n} \binom{6}{n}}{\binom{49}{6}}$$

- Probability of guessing k AND the complementary

$$\Pr(n) = \frac{\binom{43}{6-n} \binom{6}{n} \binom{43-n}{1}}{\binom{49}{6}} = \frac{\binom{43}{6-n} \binom{6}{n}}{\binom{49}{6}} (43-n)$$



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## Combinatorial Methods.Lotto6/49

- Probability of guessing k AND NOT the complementary

$$\Pr(n) = \frac{\binom{43}{6-n} \binom{6}{n} \binom{43-(6-n)}{1}}{\binom{49}{6}} = \frac{\binom{43}{6-n} \binom{6}{n}}{\binom{49}{6}} (37+n)$$



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## Combinatorial Methods. Coincidences

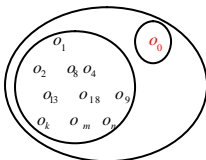
- In a planet with years of N days, we have n habitants n < N.
  - Probability that at least two have the birthday at the same day.
  - Value of n in so that the probability of coincidence is 1/2
    - Note: At-least-one=one OR two OR....

Example taken from VÉLEZ, HERNÁNDEZ, Cálculo de Probabilidades

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## Combinatorial Methods. Coincidences

- Note:
  - $A = \{\text{At-least-one}\} = \{\text{one OR two OR....}\}$
  - $\Pr(\Omega - \{a_0\}) = \Pr(a_1) + \Pr(a_2) + \Pr(a_3) + \dots = 1 - \Pr(a_0)$



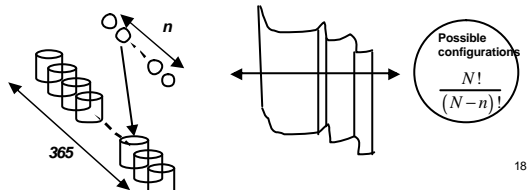
**Strategy** We will compute the probability that there is no coincidence

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## Combinatorial Methods. Coincidences

- Possible ways of having no coincidence

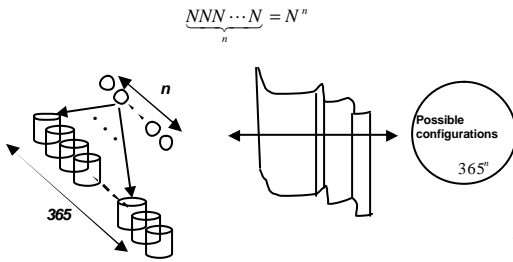
$$\underbrace{365}_{N} \underbrace{364}_{N-1} \underbrace{363}_{N-2} \dots \underbrace{365-(n-2)}_{N-(n-2)} \underbrace{365-(n-1)}_{N-(n-1)}$$



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## Combinatorial Methods. Coincidences

- Possible ways of allocation of birthdays



## Combinatorial Methods. Coincidences

- Final probability:

$$\Pr(A) = 1 - \frac{N!}{(N-n)! N^n}$$

n	5	10	15	20	25	50
p	0.027	0.117	0.253	0.411	0.569	0.97

- From Paulo's Book:
  - TV show, the presenter bets that someone in the public (>50) has the same birthday.
  - What is the mistake?

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## Combinatorial Methods. Coincidences

- Final probability:

$$\Pr(A) = 1 - \frac{N!}{(N-n)! N^n}$$

– Can be difficult to compute.

– We substitute by the Stirling's aprox.  $N! \approx \sqrt{2\pi N} N^{N+1/2} e^{-N}$

$$1 - \Pr(A) = \frac{\sqrt{2\pi} N^{N+1/2} e^{-N}}{\sqrt{2\pi} (N-n)^{N-n+1/2} e^{-(N-n)} N^n} = \left(1 - \frac{n}{N}\right)^{n-1/2-N} e^{-n}$$

$$\lim_{a \rightarrow \infty} \left(1 + \frac{x}{a}\right)^a = e^x$$

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## Combinatorial Methods. Coincidences

- Final probability:  $\Pr(A) = 1 - \frac{N!}{(N-n)! N^n}$

$$1 - \Pr(A) = \left(1 - \frac{n}{N}\right)^{n-1/2-N} e^{-n}$$

$$(n-1/2-N) \log\left(1 - \frac{n}{N}\right) \cong (n-1/2-N) \left(-\frac{n}{N} - \frac{n^2}{2N^2}\right)$$

$$\log(1+p) \cong -p - \frac{p^2}{2} - \frac{p^3}{3} - \dots$$

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## Combinatorial Methods. Coincidences

- Final probability:  $\Pr(A) = 1 - \frac{N!}{(N-n)! N^n}$

$$(n-1/2-N) \log\left(1 - \frac{n}{N}\right) \cong (n-1/2-N) \left(-\frac{n}{N} - \frac{n^2}{2N^2}\right) \cong n - \frac{n(n-1)}{2N}$$

$$1 - \Pr(A) = \left(1 - \frac{n}{N}\right)^{n-1/2-N} e^{-n} \cong e^{-\frac{n(n-1)}{2N}}$$

When we study Poisson we will get the same expression ☺

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## Combinatorial Methods

- The movie "The Quick and the Dirty" a russian roulette duel, with 6 identical shots of whiskey, one is laced with Strychnine.
- Two duelist must drink at turns.
- Bad guy offers \$1000 to the good if he drinks first.
- **Is it worth while?**

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## Combinatorial Methods

- Possible arrangements of the glasses=6!
- Possible arrangements if the strychnine glass is the first one=5!
- P(First Glass has Stry.)=5!/6!=1/6
  - Notice that by symmetry, all have the same prob.
- Prob(A duelist drinks Stry)

$$\Pr(A) = \frac{3 \cdot 5!}{6!} = \frac{1}{2}$$

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## Combinatorial Methods

- Prob. that any duelist drinks Strychnine

$$\Pr(A) = \frac{3 \cdot 5!}{6!} = \frac{1}{2}$$

- If the first survives, the prob. that the bad one drinks it is:

$$\Pr(A) = \frac{3 \cdot 4!}{5!} = \frac{3}{5}$$

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## Combinatorial Methods “socks problem” (*Murphy’s law*)

- The “socks problem” (*Murphy’s law*)
  - Take 10 different pairs of socks to the laundromat.
  - What are the probabilities of seven and four matching socks remaining when 6 socks are lost during the washing.?
  - Best scenario 7 matching pairs. Worst case 4 matching pairs.
    - Does it look much different?

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## Combinatorial Methods “socks problem” (*Murphy’s law*)

- Possible ways to choose 6 socks of 10 pairs.
 
$$\binom{20}{6}$$
- We have 7 complete pairs only if both socks of the three pairs are missing which can happen in  $\binom{10}{3}$  ways
- Final probability:
 
$$\Pr = \frac{\binom{10}{3}}{\binom{20}{6}} = 0.0031$$

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## Combinatorial Methods “socks problem” (*Murphy’s law*)

- You are left with **only** 4 matching pairs.
- If only exactly one sock of each of the 6 pairs is missing, the ways for choosing 6 socks so that 4 matching socks are left is

$$\binom{10}{6} 2^6 \leftarrow \{s_1^i, s_2^i\}$$

- Final probability:
 
$$\Pr = \frac{\binom{10}{6} 2^6}{\binom{20}{6}} = 0.34$$

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## Combinatorial Methods “The neighbour problem”

- Eight important heads of state, the US president and the British prime minister are present at a summit. At the photo the dignitaries are lined up randomly.
- Compute the probability that they are next to each other.

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