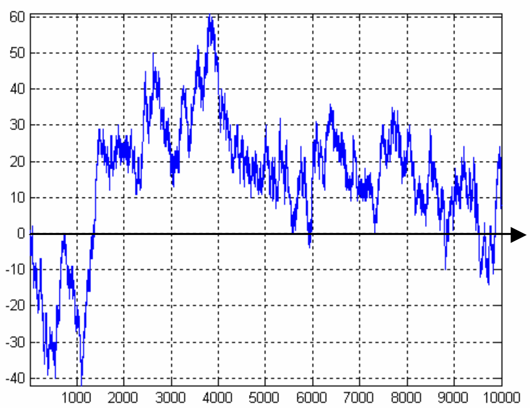
Random Variables

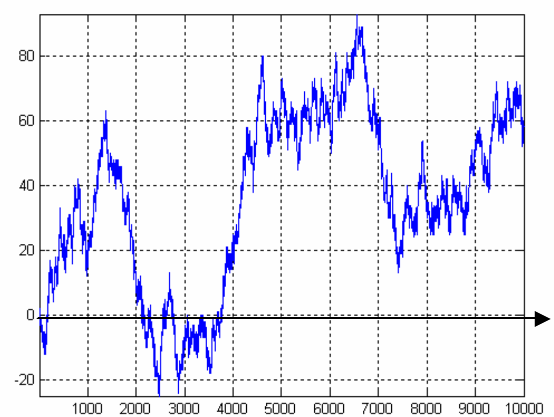
What is a random Variable?

- RV=***Experiment***+***Measure*** of interest
  - Roullette:
  - ***Experiment***
    - Roll the ball (rien ne va plus)
  - ***Measure***
    - A={A red has appeared}
    - B ={ An Odd number has appeared}
    - C ={ House winns}

What is a random Variable?

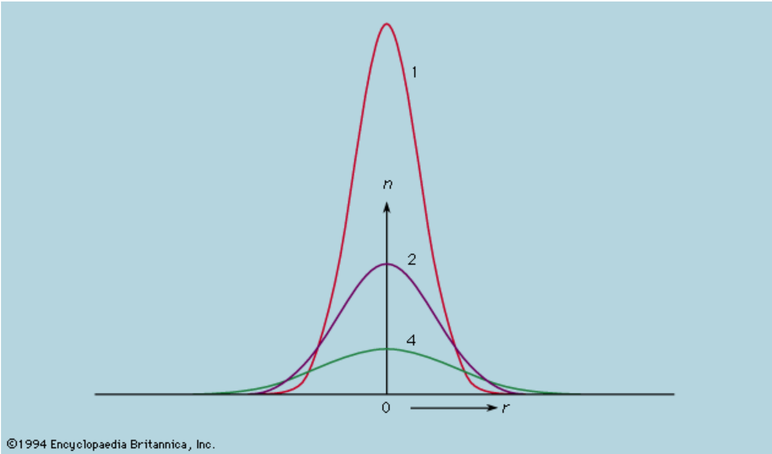
- ***Experiment***
  - Flip coins: Heads->+1,Tail->-1
- ***Measure***
  - A ={Fraction of time one is winning}
  - B ={Distance between zero crossings}
  - C ={Maximal distance to the zero point}

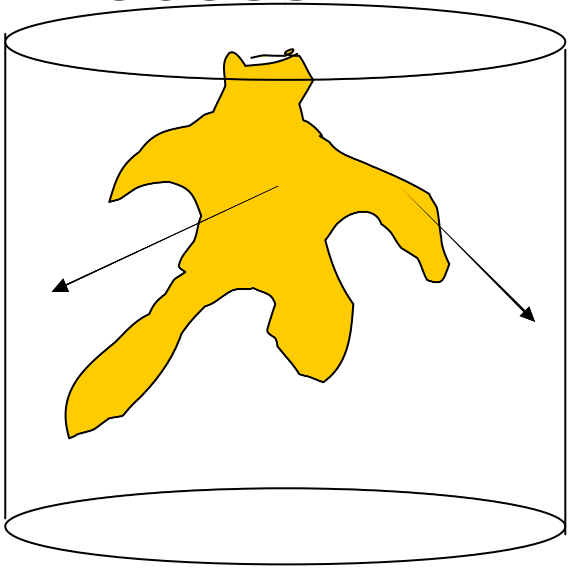


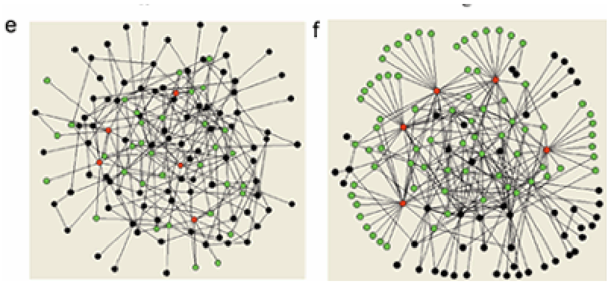


What is a random Variable?

- ***Experiment*** :Diffusion Process







Propagation of messages in a net

- ***Measure***

A ={Maximal distance as a function of time}

B={**Area/Perimeter**}

What is a random Variable?

- Definition:
- In a Probability Space (Ω,P), a Random Variable (rv), is a function:

$$X : \Omega \to \Re$$

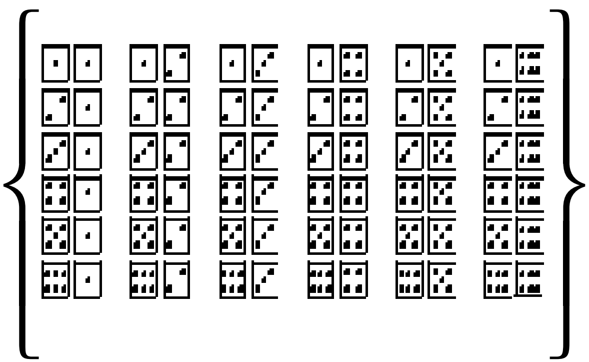
Example

- Two players (A,B) roll two dice
- Possible **random variables**:

$$X_A : \Omega \rightarrow \Re$$

$$X_A(i, j) = i / j \quad \text{for } (\text{i}, \text{j}) \in \Omega$$

$\Omega =$

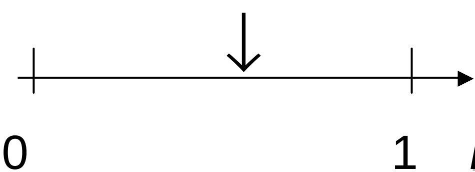


We would like to define a probability function

$$P : \Omega \rightarrow \Re$$

$$P(X_A(i, j))$$

$$X_A(i, j)$$



Example

- Two players (A,B) roll two dice
- Possible random variables:

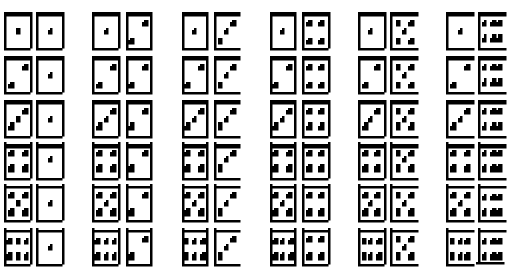
$$X_{Win} : \Omega \rightarrow \Re$$

$$X_{Win}(i, j) = \begin{cases} 1 \text{ if } i > j \\ 0 \text{ if } i = j \\ -1 \text{ if } i < j \end{cases} \quad \text{for } (i, j) \in \Omega$$

$$X_{Diff} : \Omega \rightarrow \Re$$

$$X_{Diff}(i, j) = i - j \quad \text{for } (i, j) \in$$

$$\Omega = \{$$

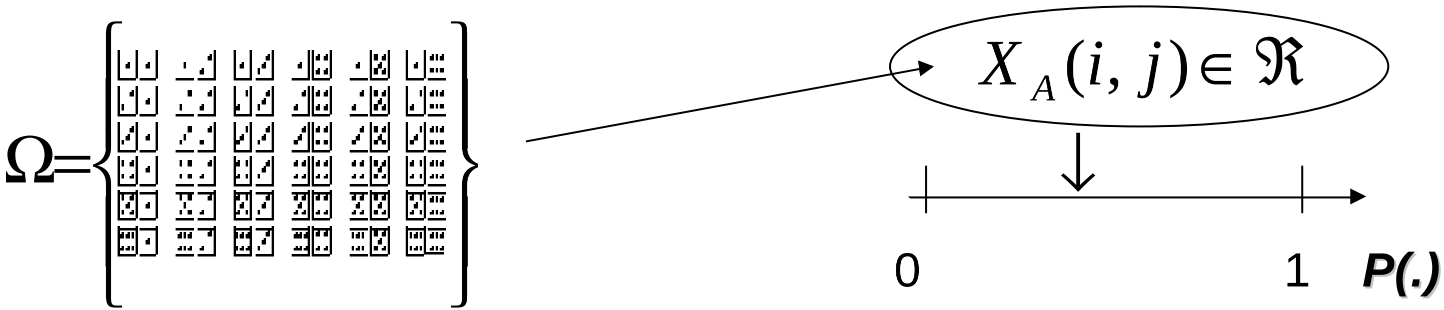


$$\}$$

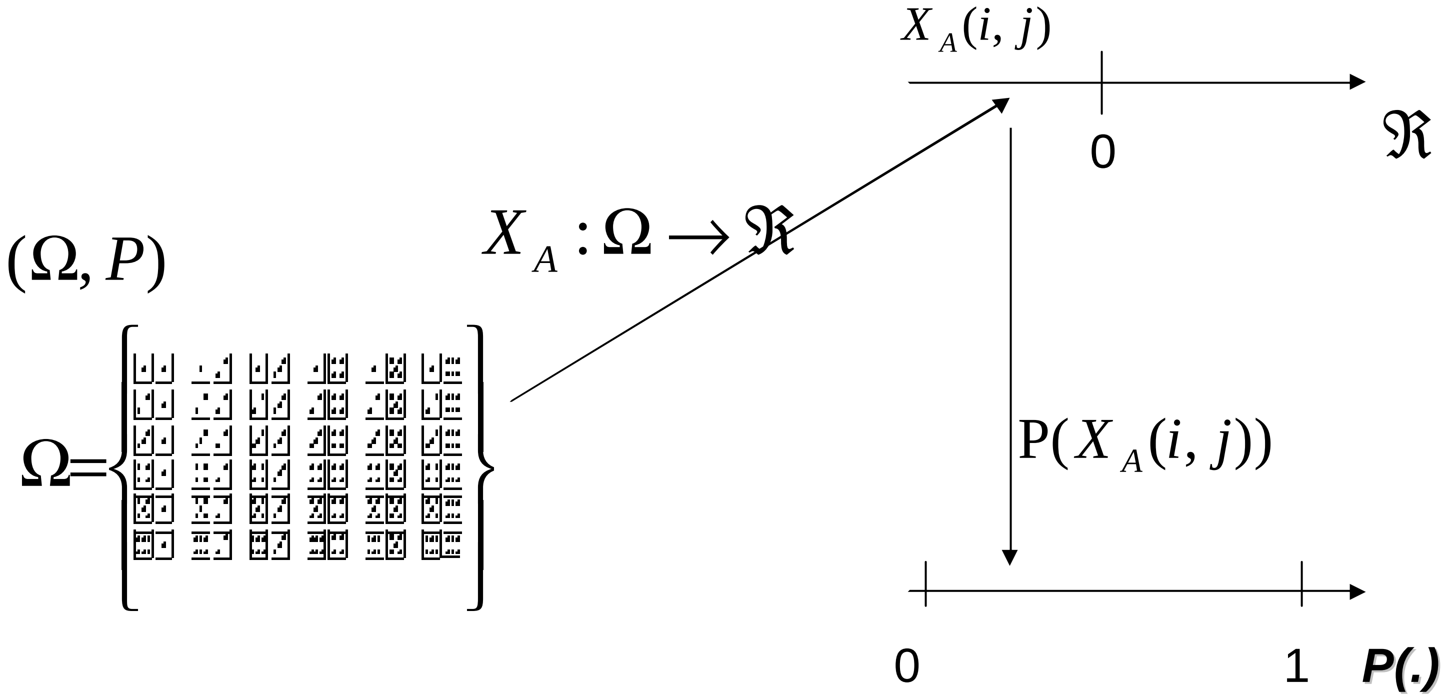
$X_{Win}(i, j)$ or $X_{Diff}(i, j)$

Objects that are needed

Name	***symbol***	***kind of object***
Sample Space	Ω	SET
Random Variable	$X_A : \Omega \rightarrow \Re$	FUNCTION ON A SET
Probability Space	(Ω, P)	SET + FUNCTION ON A SET
Prob. of a RV	$\mathrm{P}(X_A(i, j))$	FUNCTION ON A $\Re$

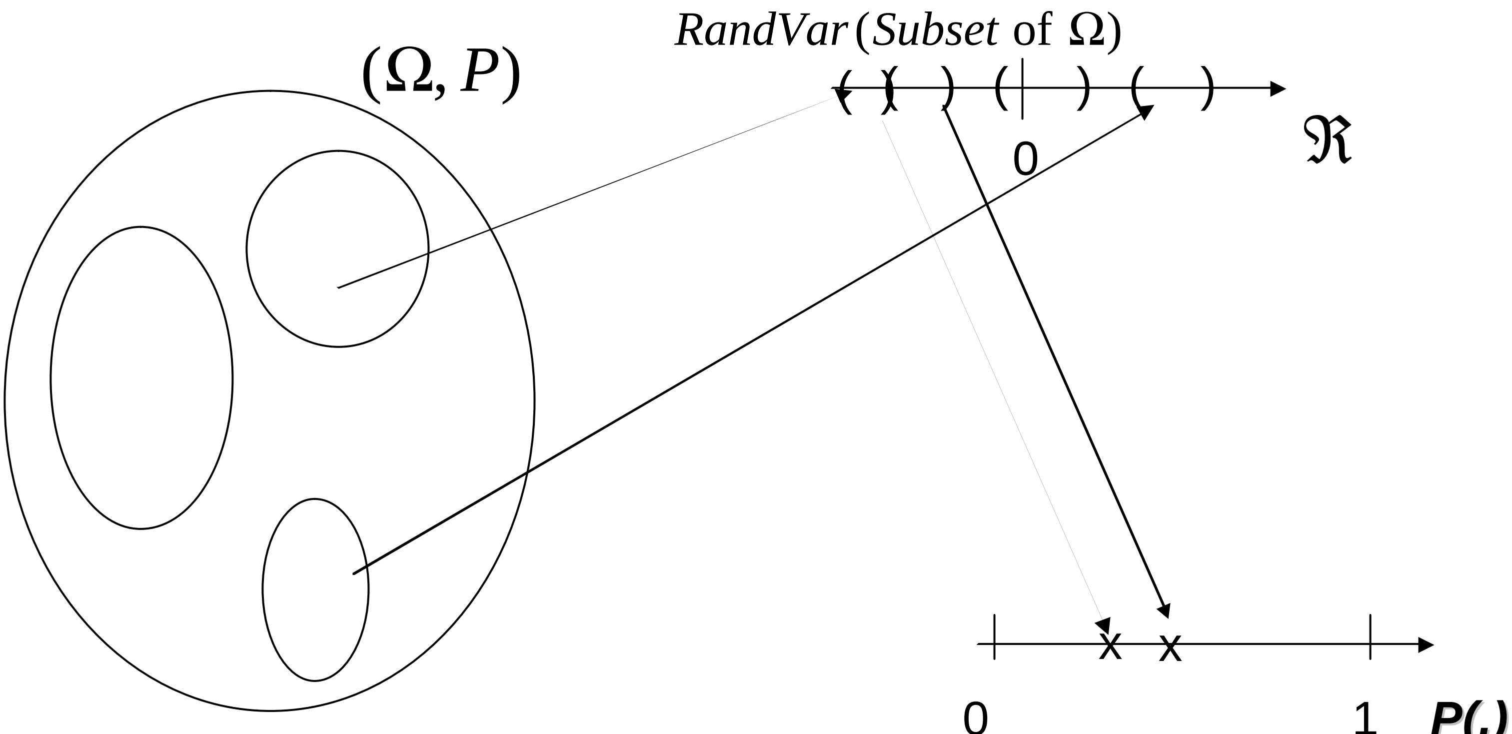


Relation between objects



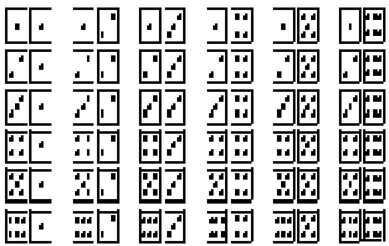
Probability of a Random Variable

- Definition: Set of probabilities associated to the values that a R.V. can take.



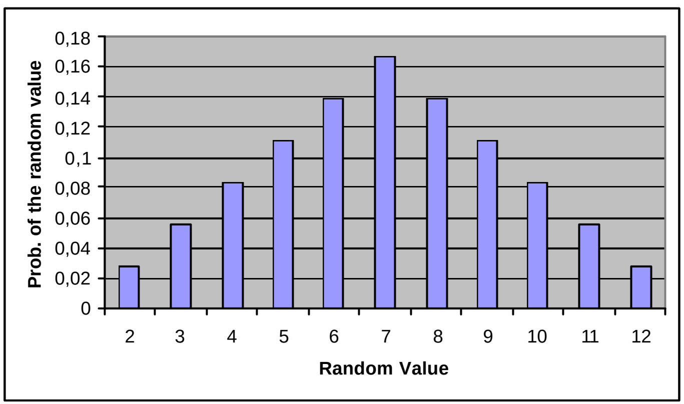
Example: Probability of a Random Variable

- A die is thrown twice

$\Omega = \{$  $\}$

$$X : \Omega \rightarrow \Re$$

$$X(i, j) = i + j \quad \text{for } (i, j) \in \Omega$$



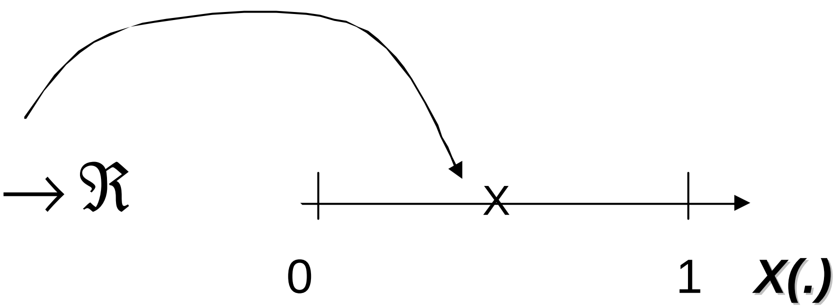
2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

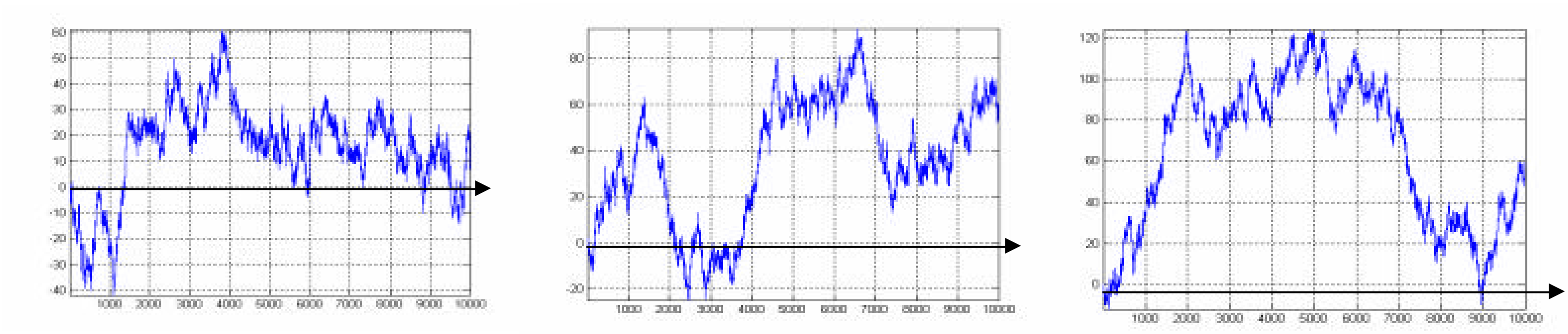
Example

- Game of Daniel and Nicolas Bernouilli
  - RV: X(HH…FFH): Fraction of time Daniel is in lead.

$$\Omega = \begin{Bmatrix} \{HHHH\cdots HHHH\} \\ \{FHHH\cdots HHHH\} \\ \vdots \\ \{HFFH\cdots FFHH\} \\ \vdots \\ \{FFFF\cdots FFFF\} \end{Bmatrix}$$

$$X:\Omega \to \Re$$





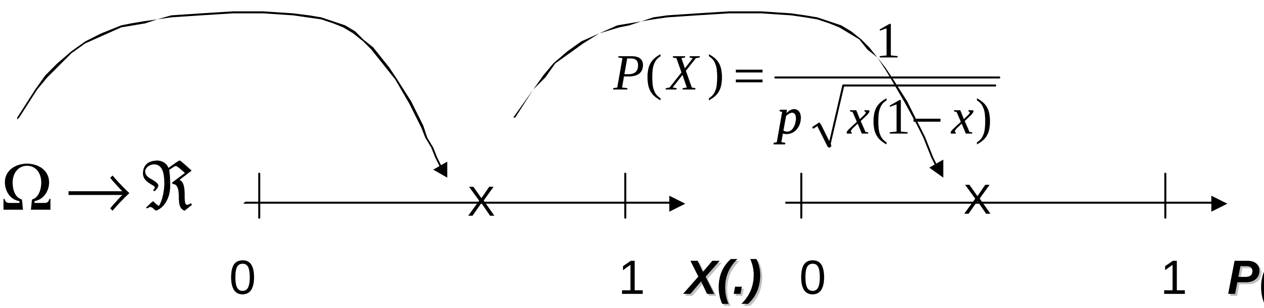
Example

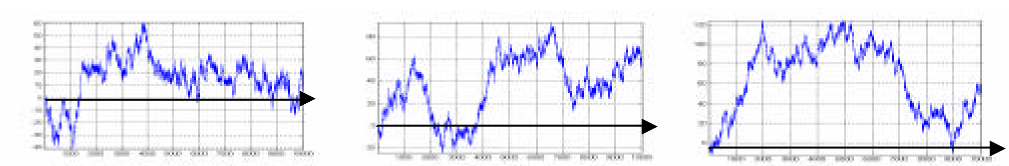
- Probability of X(*HH…FFH*):
  - We compute P(X(*HH…FFH*)) for each $A \in \Omega$

$$\Omega = \left\{ \begin{array}{c} \{HHHH \cdots HHHH\} \\ \{FHHH \cdots HHHH\} \\ \vdots \\ \{HFFH \cdots FFHH\} \\ \vdots \\ \{FFFF \cdots FFFF\} \end{array} \right\}$$

$$X : \Omega \rightarrow \Re$$

$$P(X) = \frac{1}{p\sqrt{x(1-x)}}$$

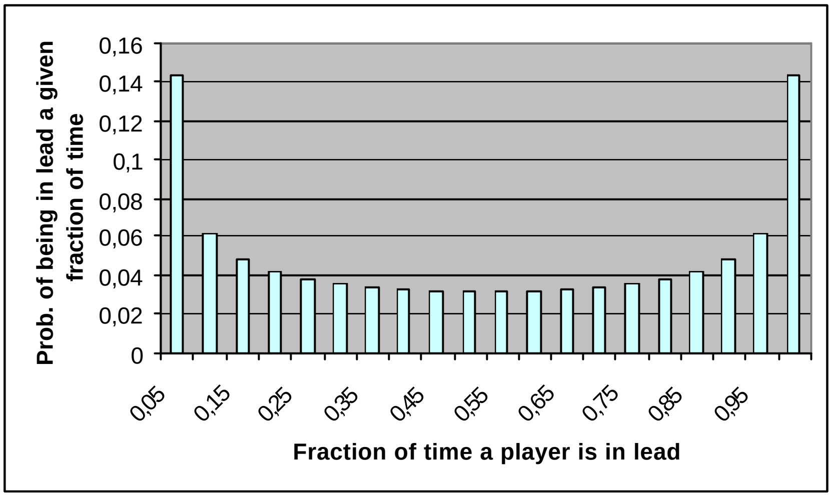




For the derivation of P(X) see Feller or Vélez

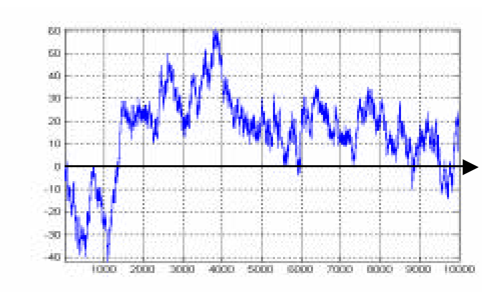
Example

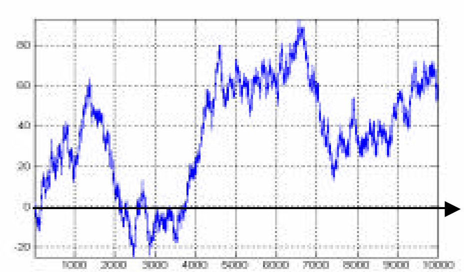
- Probability of X(HH…FFH):
  - RV: Fraction of time Daniel is in lead.

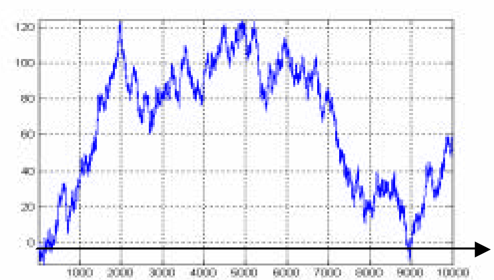


$$X : \Omega \rightarrow \Re$$

$$P(X) = \frac{1}{p\sqrt{x(1-x)}}$$

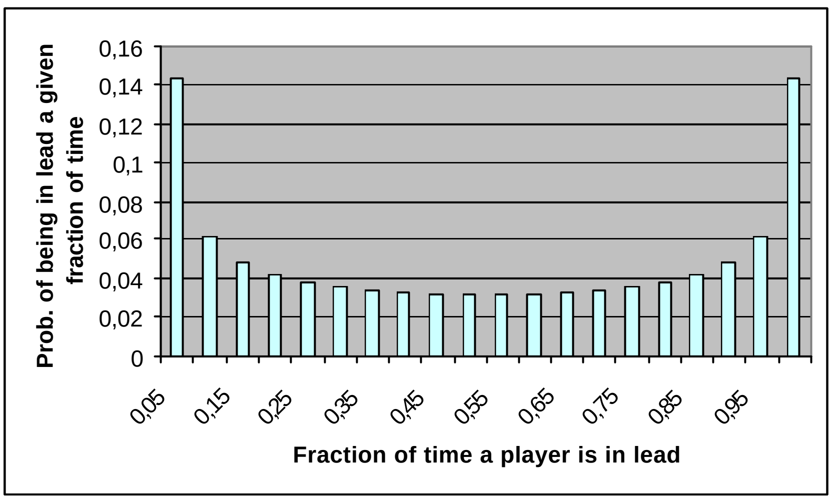






Example

- Probability of X(HH…FFH):
  - RV: Fraction of time Daniel is in lead.
    - Note that the distribution says that Daniel with high probability either wins of losses

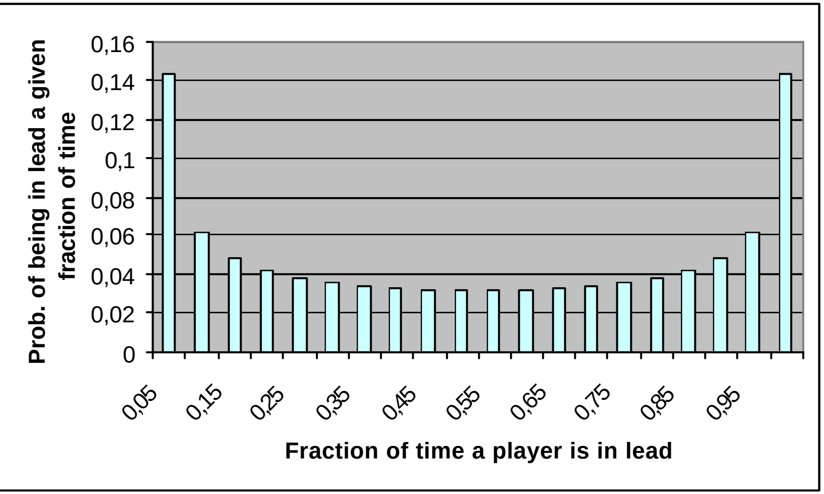


$$X:\Omega \rightarrow \Re$$

$$P(X) = \frac{1}{p\sqrt{x(1-x)}}$$

Example

- Probability of X(HH…FFH):
  - Underlying mechanism.
  - In 20 throws the probability of a run of 5 consecutive heads or tails is much higher than intuition says

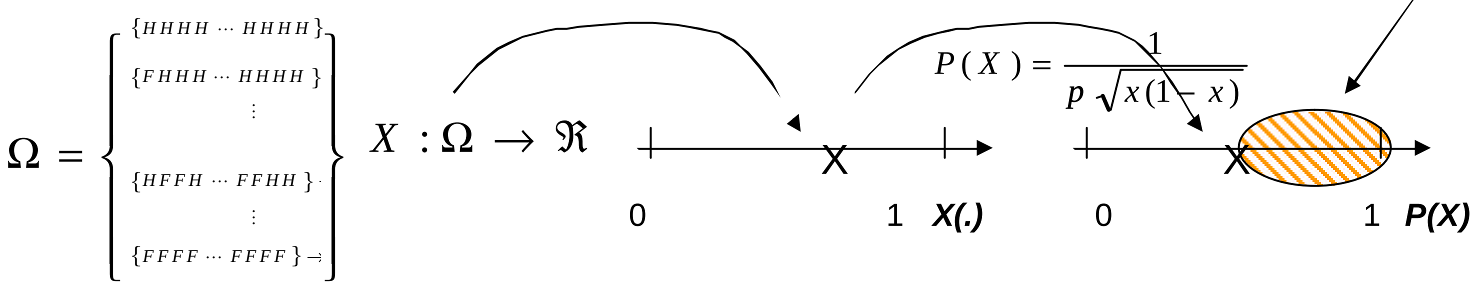


•Hot hand in basquetball
•Highly succesfull wall street trade
•Random numbers test.
Methods for Roullete
Shannon’s coin game

Example

- New R.V. Y: Daniel is in lead more than a given fraction of time.
  - Meaning: Y={X>f}

$$P(X > f) = \int_f^1 \frac{1}{p\sqrt{x(1-x)}}dx$$



$$Y : \Omega \rightarrow \Re$$

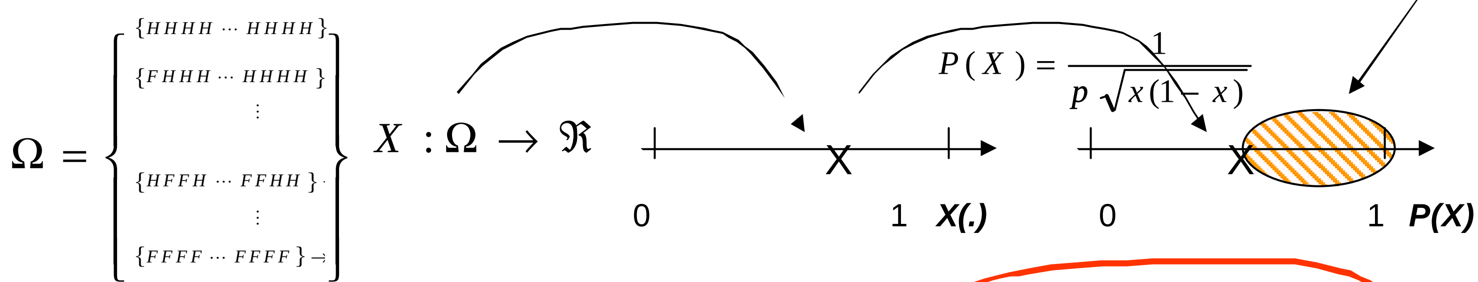
$$\Omega \rightarrow X \rightarrow Y$$

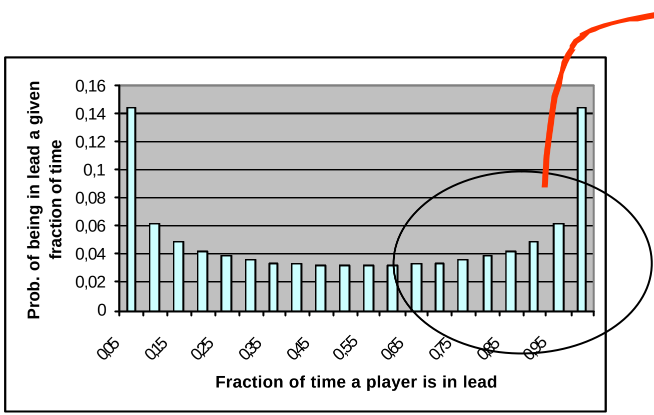
$$P(Y) = P(X > f) = \int_f^1 \frac{1}{p\sqrt{x(1-x)}}dx \approx 1 - \frac{2}{p}\arcsin(\sqrt{f})$$

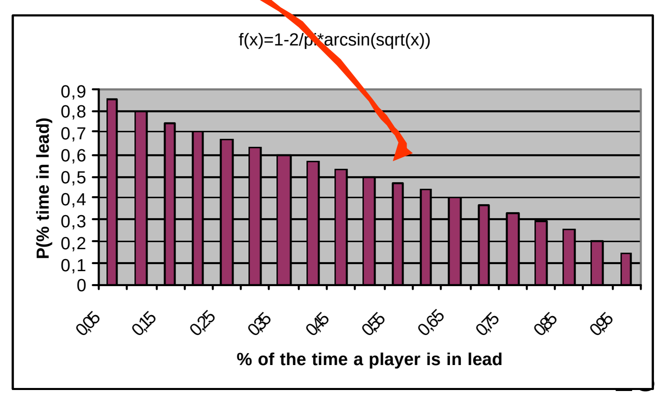
Example

- New R.V. Y: Daniel is in lead more than a given fraction of time.
  - Meaning: Y={X>f}

$$P(X > f) = \int_f^1 \frac{1}{p\sqrt{x(1-x)}}dx$$

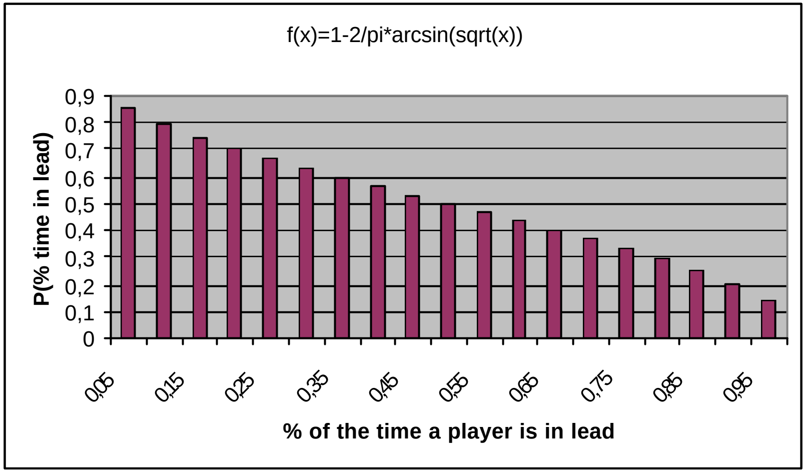






Example

- New Rand.Var. *Y*: Daniel is in lead more than a given fraction of time.
  - Meaning: Y={X>f}



$$Y : \Omega \rightarrow \Re$$

$$\Omega \rightarrow X \rightarrow Y$$

$$P(Y) = P(X > f) = \int_f^1 \frac{1}{p\sqrt{x(1-x)}} dx \approx 1 - \frac{2}{p} \arcsin(\sqrt{f})$$

Distribution of a Random Variable

- Definition:
  - The distribution function of a ***discrete*** random variable X is defined as:

$$F(x) = P(X \leq x) = \sum_{\forall x_i \leq x} P(X = x_i)$$

  - With

$$X : \Omega \rightarrow \Re$$

$$\Omega = \{A_1, A_2, \cdots A_n\}$$

$$\downarrow$$

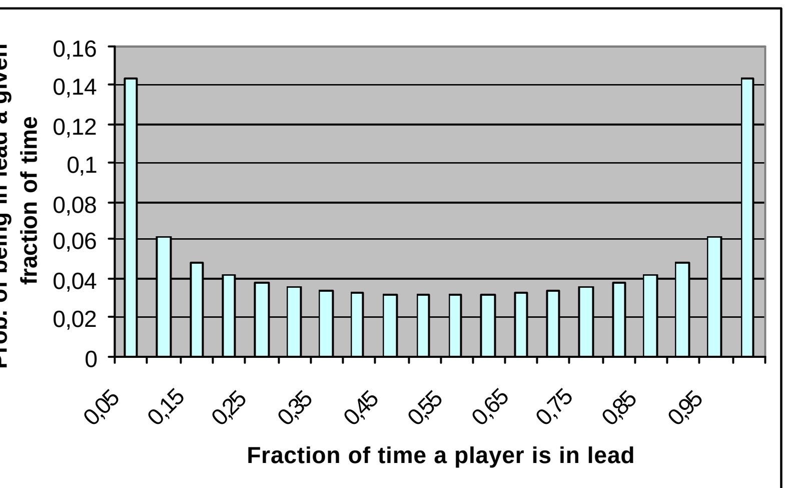
$$X = \{x_1, x_2, \cdots x_m\}$$

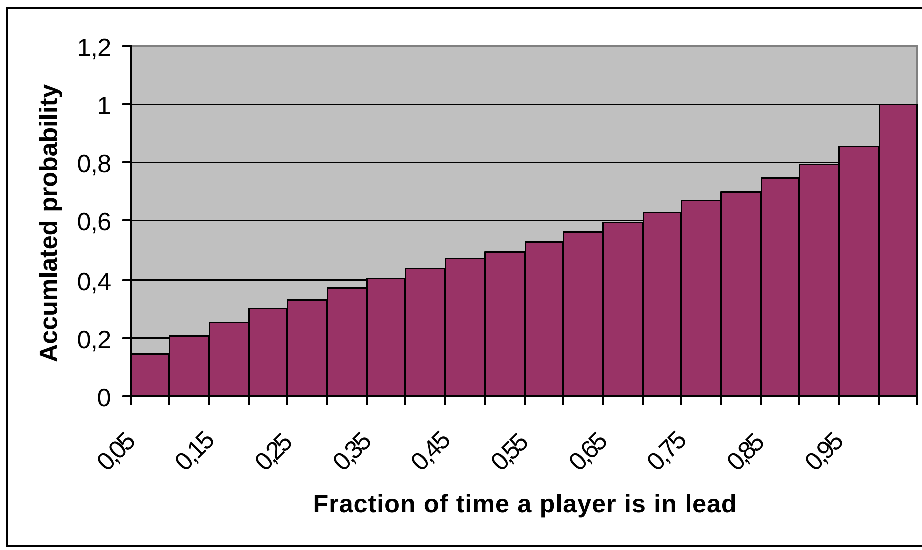
Example

- Distribution function of the Rand.Var.: Fraction of time Daniel is in lead

$$X : \Omega \rightarrow \Re$$

$$F(x) = P(X \leq x) = \sum_{\forall\, x_i \leq x} P(X = x_i) = \sum_{\forall\, x_i \leq x} \frac{1}{p\sqrt{x_i(1-x_i)}}$$

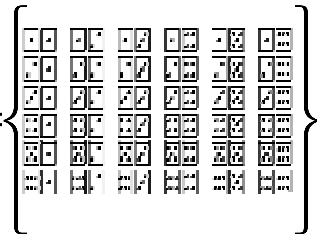




Example

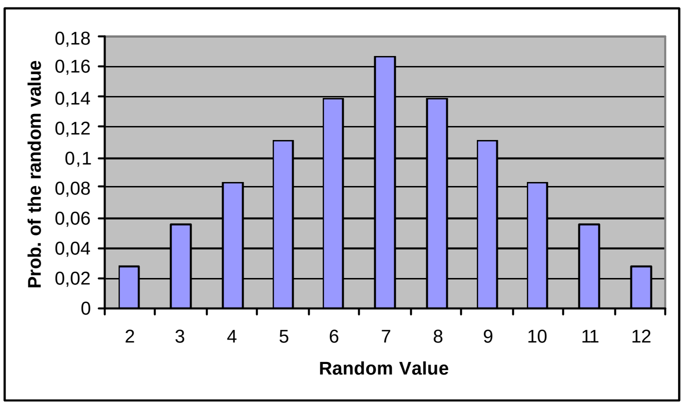
- Distribution function of the Rand.Var. : sum of the values when a die is thrown twice

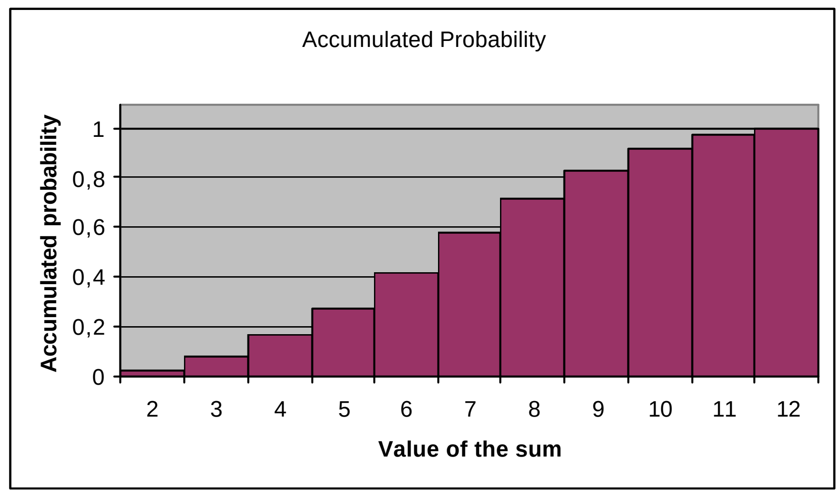
$\Omega=$



$$X:\Omega \rightarrow \Re$$

$$X(i,j) = i + j \quad \text{for } (\text{i, j}) \in \Omega$$





$$F(x) = P(X \le x) = \sum_{\forall\, x_i \le x} P(X = x_i)$$

Example

- Distribution function of a Bernouilli trial until the first success, with probability of success ***p***.
  - Model of a Bernouilli trials:
    - Flipping a coin until a head appears.
    - Trying to get a connection until the service is given.
    - Booking a place in an airplane.
  - Probability of a success in ***n*** trials

$$\Omega \to N$$

$$P(X = n) = (1 - p)^{n-1} p$$

Example

- Distribution function of a Bernouilli trial until the first success, with probability of success ***p***.

$$F(x) = P(X \le x) = \sum_{\forall\, n \le x} P(X = n) = \sum_{\forall\, n \le x} (1-p)^{n-1} p \underset{\substack{Geometric \\ Series}}{=}$$

$$= p\,\frac{1-(1-p)^n}{1-(1-p)} = 1-(1-p)^n$$

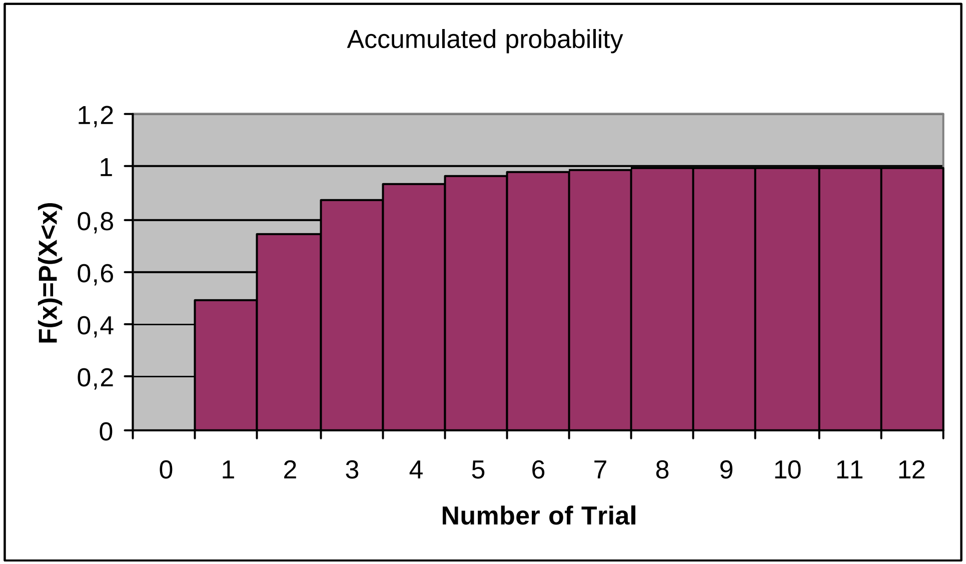
Note: Complementary of the event,={does appears in **none** of the n trials}

Example

- Distribution function of a Bernouilli trial until the first success, with probability of success ***p***.

$$F(x) = P(X \leq x) = \sum_{\forall n \leq x} P(X = n) = 1 - (1 - p)^n$$

- Coin p=1/2

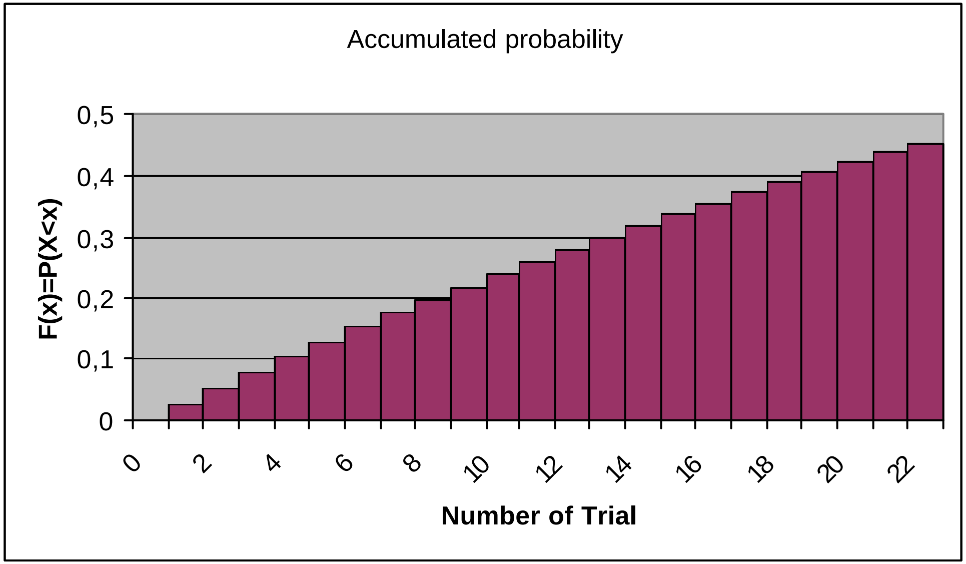


Example

- Distribution function of a Bernouilli trial until the first success, with probability of success ***p***.

$$F(x) = P(X \le x) = \sum_{\forall\, n \le x} P(X = n) = 1 - (1 - p)^n$$

- Roullette p=1/37



Example: Propagation of a rumor

- In a village of **n** inhabitants, one person explains something to a second, who tells is to a third, etc.
- The first selects one in the **n-1** remaining. From then on, all the others select between **n-2** (i.e. excludes the previous, and the original)

a) Distribution of the number of times that a rumor is transmited until it reaches again the source

b) Distribution of the number of times that a rumor is transmited until someone get’s it twice. (**Exercise**)

Taken from Cálculo de probabilidades I, R. Vélez & V. Hernández

Example: Propagation of a rumor

- Distribution of the number of times that a rumor is transmited until it reaches again the source.

$$A_0 \underset{\text{Generates the rumor}}{\longrightarrow} A_1 \underset{\text{n-2 different}}{\longrightarrow} A_2 \underset{\text{n-3 different}}{\longrightarrow} A_3 \underset{\text{n-3 different}}{\longrightarrow} A_4 \cdots$$

  - A_j (j>2) cannot select the previous or de original
  - Note that A_1 and A_2 cannot select A_0

Example: Propagation of a rumor

- Distribution of the number of times that a rumor is transmited until it reaches again the origin.
- We define the random variable:
  - R: Number of times that the rumor is transmited until it returns to A_0
- Scheme of the solution:

$$P(R=r) = P(R > r-1) - P(R > r)$$

Example: Propagation of a rumor

- We will compute first $P(R > r)$
- Remember:

$$A_0 \underset{\substack{\text{Generates} \\ \text{the rumor}}}{\rightarrow} A_1 \underset{\substack{\text{n-2} \\ \text{different}}}{\rightarrow} A_2 \underset{\substack{\text{n-3} \\ \text{different}}}{\rightarrow} A_3 \underset{\substack{\text{n-3} \\ \text{different}}}{\rightarrow} A_4 \cdots \underset{\substack{\text{n-3} \\ \text{different}}}{\rightarrow} A_r \underset{\substack{\text{n-3} \\ \text{different}}}{\rightarrow} A_{r+1} \underset{\substack{\text{n-2} \\ \text{different}}}{\rightarrow} \cdots$$

  - A_j (j>2) cannot select the previous or de original
  - Note that A_1 and A_2 cannot select A_0
    - A_1 and A_2 do not count. A_1 still has to talk to someone
- Event {R>r}

$$\{R > r\} = \{A_3, A_4 \cdots, A_r \neq A_0\} AND \{\text{Any other sequence}\}$$

$$P(R > r) = \left(\frac{n-3}{n-2}\right)^{r-2}$$

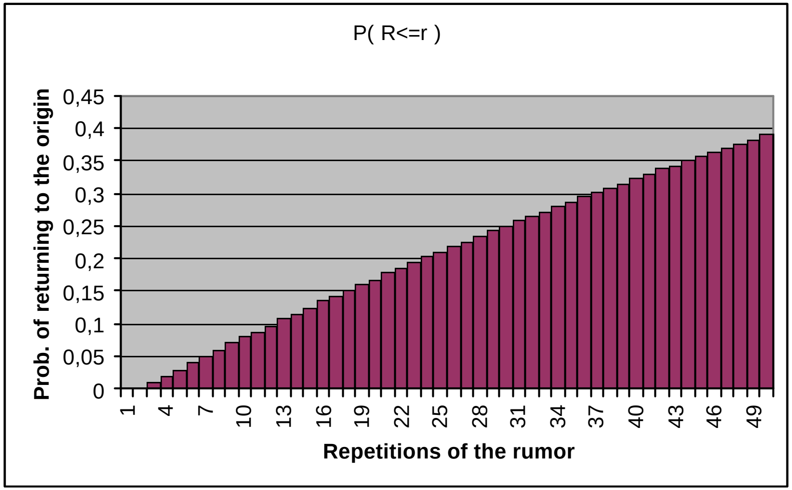
Example: Propagation of a rumor

- Distribution function of R

$$F(r) = P(R \leq r) = 1 - P(R > r)$$

$$F(r) = \begin{cases} 0 & \text{if } r < 3 \\ 1 - \left(\dfrac{n-3}{n-2} \right)^{r-2} & \text{if } r \geq 3 \end{cases}$$

In a department with **n=100**



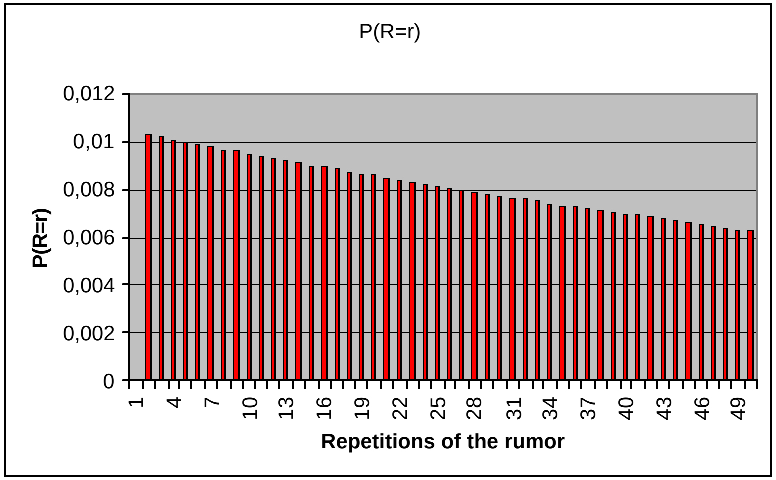
Example: Propagation of a rumor

- Probability of a given ***r***

$$P(R=r)=P(R>r-1)-P(R>r)$$

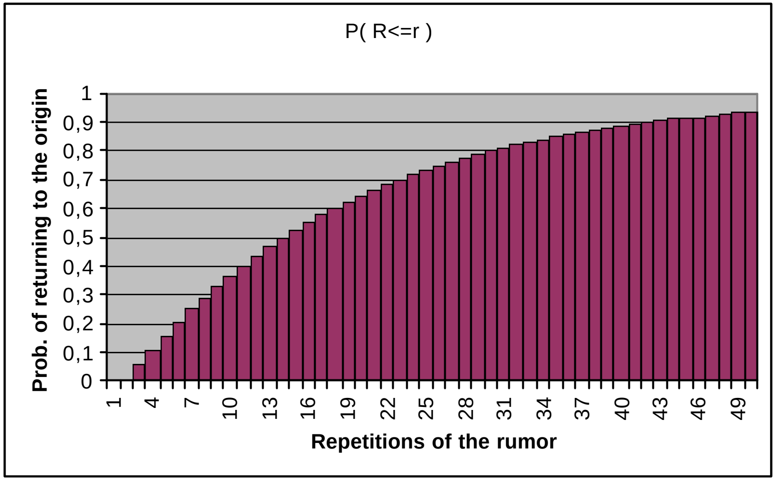
$$P(R=r)=\left(\frac{n-3}{n-2}\right)^{r-3} - \left(\frac{n-3}{n-2}\right)^{r-2} = \left(\frac{n-3}{n-2}\right)^{r-3}\frac{1}{n-2}$$

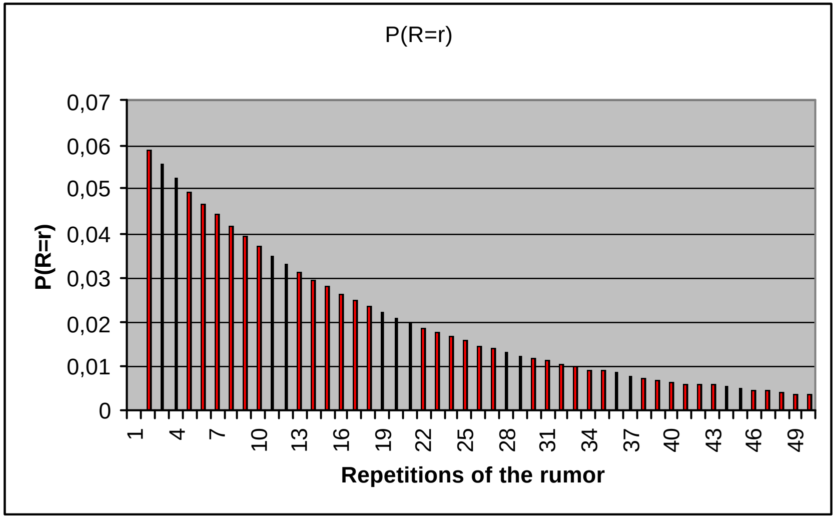
In a department with **n=100**



Example: Propagation of a rumor

- With **n=20**





- Note that there is the possibility of *loops*