Problems of Allocation Of Probabilities

Combinatorial Methods
“The neighbour problem”
• Eight important heads of state, the US president and the British prime minister are present at a summit. At the photo the dignitaries are lined up randomly.
• Compute the probability that they are next to each other.

• Number of heads \( \{ h_1, h_2, \cdots, h_8 \} \)
  \( h_1 \rightarrow US \) \( h_2 \rightarrow UK \)
• Possible arrangements \( 8! \)
• If \( \{ h_i, h_{i+1} \} \) are fixed, remains \( 6! \) arrangements

• Condition of being next to each other
• Two possibilities: \( \{ h_i, h_{i+1} \} = \{ h_i, h_{i+1} \} \)
• Possible arrangements: \( 6! \times 2 \)
• Probability \( \Pr = \frac{6! \times 7 \times 2}{8!} = \frac{1}{4} \)

Another Approximation for the Birthday problem

• Derivation:
\[
\Pr(A) = 1 - \prod_{i=1}^{N} \left( 1 - \frac{1}{N} \right) = 1 - \prod_{i=1}^{N} \left( 1 - \frac{1}{i} \right) = 1 - \frac{1}{N^{N}} = 1 - e^{-N/N^{2}}
\]

Another Approximation for the Birthday problem

• Aprox. of the exponential:
\[
e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \cong 1 + x
\]

\[\text{Worst case} \quad 23/365=0.0630\]
Another Approximation for the Birthday problem

• Arithmetic sum:

\[ S = \sum_{i=1}^{n} \left( \frac{1}{i(i+1)(i+2)} \right) \]

A solution to the secretary problem

• Taken from T. Ferguson, Who solved the secretary problem?

• Assumptions:
  1. There is one secretarial position available.
  2. The number of applicants \( n \) is known.
  3. The applicants are interviewed sequentially in random order.
  4. Decision is made on relative ranks.
  5. Rejected applicant cannot be recalled.
  6. Payoff is 1 if you choose the best and 0 otherwise.

A solution to the secretary problem

• Empirical simulations:
  – Case of \( n=4 \)

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A solution to the secretary problem

• Empirical simulations:
  – Case of \( n=5 \) (120)

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A solution to the secretary problem

• Empirical simulations:
  – Case of \( n=6 \) (720)

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A solution to the secretary problem

• Condition for solving the problem:
  – The best is at position \( j \), and before \( r \) we have the second best.
  • Note: Relative ranking to the ones that have been examined. Absolute values do not count.
  • Absolute value
    – Should be treated equal
  • Relative ranking
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A solution to the secretary problem

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A solution to the secretary problem

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A solution to the secretary problem

- Formal description of the problem:
  
  \[ P(\text{the second best until j is before } r) = \frac{(n-1)!}{n^r} \]

- An approximation if the sum is big enough
  
  \[ \text{We substitute the sum by the integral} \]

\[ P(r) = \sum_{j=1}^{n-1} P(j \text{ is best before } r) \approx \frac{1}{n} \int_{1}^{n-1} \log \left( \frac{n}{j} \right) \, dj \]

Appendix. Integral of a logarithm

\[ \int_{a}^{b} \log(x) \, dx = \log(b) - \log(a) \]
A solution to the secretary problem

• How good is the approximation?
  - $N=50$, $r=37\%$ and $r=15\%$

\[
Pr(r) = \frac{1}{n} \sum_{j=1}^{n} \frac{e^{-j/n}}{j/n}
\]

Urn Problem

• $n$ balls are randomly distributed into $n$ urns. Compute the probability that:
  1. A given urn is empty
  2. At least one urn is empty
  3. A given urn is the only empty
  4. There is only one empty urn

Urn Problem

• A given urn is empty
  - The balls can be distributed into $(n-1)$ urns leaving a given one empty.

\[
Pr(A \text{ given urn}) = \frac{(n-1)!}{n^n}
\]

Urn Problem

• At least one urn is empty
  - We should compute the complementary probability

\[
Pr(\Omega - \{\sigma_n\}) = Pr(\sigma_1) + Pr(\sigma_2) + Pr(\sigma_3) + \cdots + 1 - Pr(\sigma_n)
\]

- There are $n!$ ways of assigning one ball to one urn

\[
Pr(\text{At least one urn}) = 1 - \frac{n!}{n^n}
\]

Urn Problem

• A given urn is the only empty
  - All the others except one have only one ball.
  - Ways of selecting a couple of balls
  - Ways of distributing the balls: $(n-1)!$

• Note that if the two balls are substituted by a new ball with different colour we have an equivalent problem.

\[
Pr = \binom{n}{2} \frac{(n-1)!}{n^n} = \frac{n(n-1)}{2n^n}
\]

Urn Problem

• There is only one empty urn
  - It is the probability that the empty urn is either the first or the second or etc.

\[
Pr(\sigma_1) + Pr(\sigma_2) + \cdots + Pr(\sigma_n)
\]

\[
Pr = n \binom{n}{2} \frac{(n-1)!}{n^n} = \binom{n}{2} \frac{n(n-1)!}{2n^n}
\]
An Election Problem (Markov)

• In an election with two candidates
  – A gets n votes
  – B gets m votes
• Compute the probability of the event:
  – E={A is always ahead in the count of votes}