Problems of
Allocation Of Probabilities

Combinatorial Methods
“The neighbour problem”

- Eight important heads of state, the US president and the British prime minister are present at a summit. At the photo the dignataries are lined up randomly.
- Compute the probability that they are next to each other.

Combinatorial Methods
"The neighbour problem"

- Number of heads $\{h_1, h_2, \cdots, h_8\}$

  $h_1 \to US \quad h_2 \to UK$
- Possible arrangements 8!
- If $\{h_1, h_2\}$ are fixed, remains 6! arrangements

Combinatorial Methods
"The neighbour problem"

- Condition of being next to each other

$\{h_1, h_2\} = \{i, i+1\}$ for $1 \le i \le 7$

- Two possibilities: $\{h_1, h_2\} = \{h_2, h_1\}$
- Possible arrangements: 6!*7*2
- Probability $$\Pr = \frac{6!*7*2}{8!} = \frac{1}{4}$$

Another Approximation for the Birthday problem

- Derivation:

$$\Pr(A) = 1 - \frac{N!}{(N-n)!}\frac{1}{N^n} = 1 - \prod_{i=1}^{n} \frac{N-i+1}{N}$$

$$= 1 - \prod_{i=1}^{n}\left(1 - \frac{i-1}{N}\right) \geq 1 - \prod_{i=1}^{n}\left(e^{-\frac{i-1}{N}}\right) = 1 - e^{-\sum_{i=1}^{n}\frac{i-1}{N}} = 1 - e^{-N(N-1)/2}$$

Aprox. of the exponential

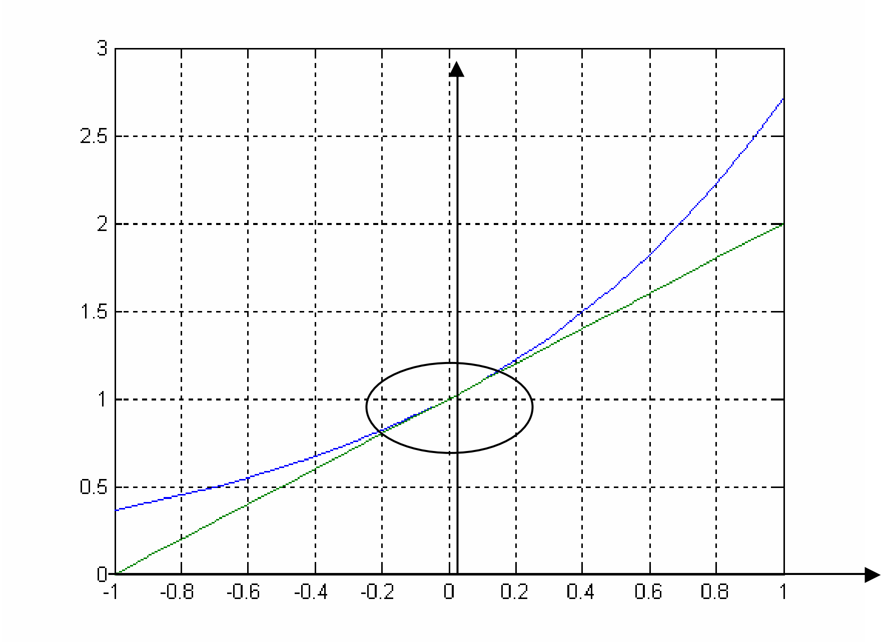
Arithmetic sum

Another Approximation for the Birthday problem

- Aprox. of the exponential:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \cong 1 + x$$

Worst case
23/365=0.0630



Another Approximation for the Birthday problem

- Arithmetic sum:

$$S = \sum_{i=1}^{n} i = 1+2+3+4+\cdots+(n-2)+(n-1)+(n)$$

$$S = \sum_{i=1}^{n} i = (n)+(n-1)+(n-2)+\cdots+4+3+2+1$$

$$2S = \sum_{i=1}^{n} i = (n+1)+(n+1)+\cdots+(n+1)+(n+1) = (n+1)n$$

$$S = \sum_{i=1}^{n} i = \frac{(n+1)n}{2}$$

A solution to the secretary problem

- Taken from T.Ferguson, Who solved the secretary problem?
- Assumptions:
  1. There is one secretarial position available.
  2. The number of applicants ***n*** is known
  3. The applicants are interviewed sequentially in random order.
  4. Decision is made on relative ranks
  5. Rejected applicant cannot be recalled
  6. Payoff is *1* if you choose the best and ***0*** otherwise

A solution to the secretary problem

- Empirical simulations:
  - Case of ***n=4***

0.25	**0.50**	**0.33**	**0.25**

Permutation

				k=1	***k=2***	***k=3***	***k=4***
4	*3*	*2*	*1*	1	0	0	0
4	*3*	*1*	*2*	1	0	0	0
4	*2*	*3*	*1*	1	0	0	0
4	*2*	*1*	*3*	1	0	0	0
4	*1*	*2*	*3*	1	0	0	0
4	*1*	*3*	*2*	1	0	0	0
3	*4*	*2*	*1*	0	1	0	0
3	*4*	*1*	*2*	0	1	0	0
3	*2*	*4*	*1*	0	0	1	0
3	*2*	*1*	*4*	0	0	0	1
3	*1*	*2*	*4*	0	0	0	1
3	*1*	*4*	*2*	0	0	1	0
2	*3*	*4*	*1*	0	1	1	0
2	*3*	*1*	*4*	0	1	0	1
2	*4*	*3*	*1*	0	1	0	0
2	*4*	*1*	*3*	0	1	0	0
2	*1*	*4*	*3*	0	0	1	0
2	*1*	*3*	*4*	0	0	1	1
1	*3*	*2*	*4*	0	1	0	1
1	*3*	*4*	*2*	0	1	1	0
1	*2*	*3*	*4*	0	1	1	1
1	*2*	*4*	*3*	0	1	1	0
1	*4*	*2*	*3*	0	1	0	0
1	*4*	*3*	*2*	0	1	0	0
Probability				**0.25**	**0.50**	**0.33**	**0.25**

A solution to the secretary problem

- Empirical simulations:
  - Case of ***n=5*** (120)

0.31	0.69	0.31	0.13	0.06

Permutation

					k=1	***k=2***	***k=3***	***k=4***	***k=5***
5	4	3	2	1	1	0	0	0	0
5	4	3	1	2	1	0	0	0	0
5	4	2	3	1	1	0	0	0	0
5	4	2	1	3	1	0	0	0	0
5	4	1	2	3	1	0	0	0	0
1	2	3	5	4	0	1	1	1	0
1	2	4	3	5	0	1	1	0	1
1	2	4	5	3	0	1	1	1	0
1	2	5	4	3	0	1	1	0	0
1	2	5	3	4	0	1	1	0	0
1	5	3	2	4	0	1	0	0	0
1	5	3	4	2	0	1	0	0	0
1	5	2	3	4	0	1	0	0	0
1	5	2	4	3	0	1	0	0	0
1	5	4	2	3	0	1	0	0	0
1	5	4	3	2	0	1	0	0	0
Probability					**0.31**	**0.69**	**0.31**	**0.13**	**0.06**

A solution to the secretary problem

- Empirical simulations:
  - Case of ***n=6*** (720)

0.17	**0.50**	**0.33**	**0.25**	**0.20**	**0.17**

  - Notice that 6*0.368=2.2

A solution to the secretary problem

- Condition for solving the problem:
  - The best is at possition ***j***, and before ***r*** we have the second best.
    - Note: Relative ranking to the ones that have been examined. Absolute values do not count.
    - Absolute value
      - xxxxxx3xxxxxxx1xxxxx2xxxxxx
    - Relative ranking
      - xxxxxx2xxxxxxx1xxxxxxxxxxxx

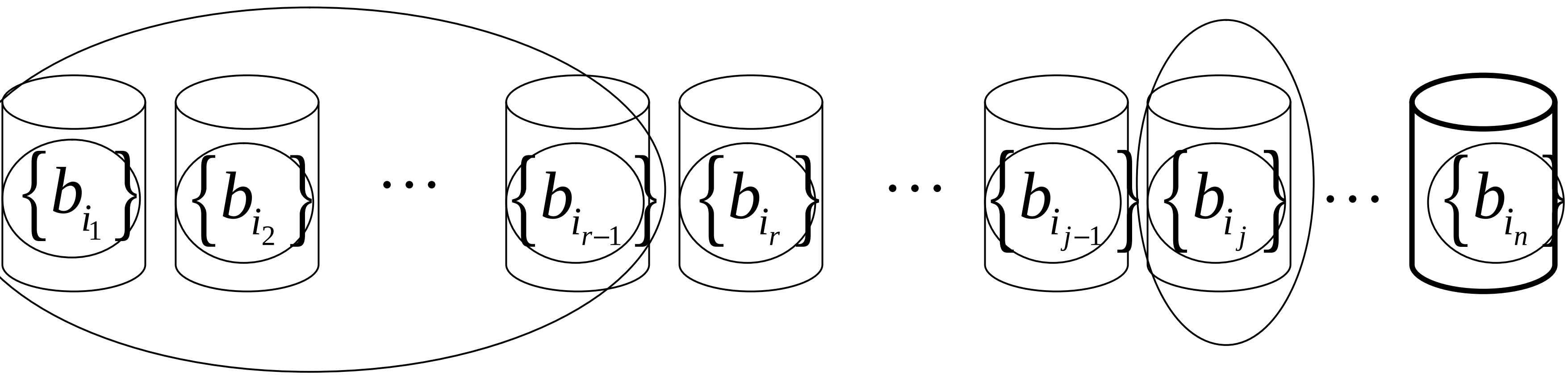
Should be treated equal

A solution to the secretary problem

- Formal description of the problem:

$$P(\textit{the}\ \text{second}\ \textit{best until}\ j\ \textit{is before}\ r) = \frac{r-1}{j-1}$$

$$P(\ j\ \textit{is best}\ ) = \frac{(n-1)!}{n!}$$



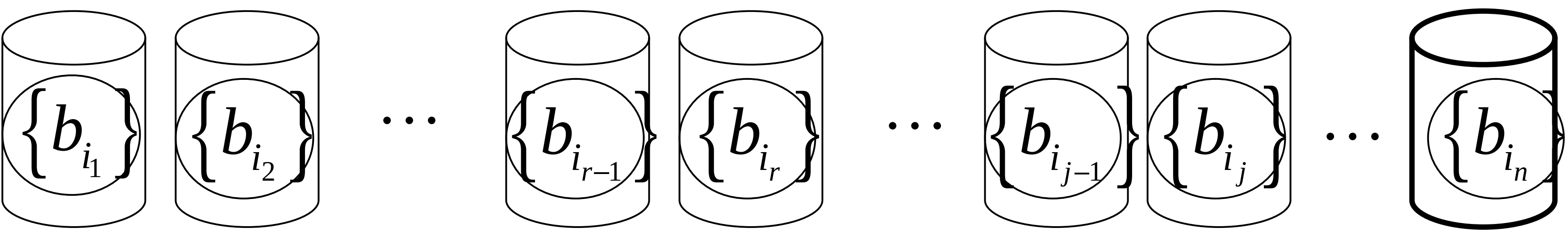
$$P(\ j\ \textit{is best}\ \textit{AND}\ \textit{before}\ r\ \textit{is an worst one}) = \frac{(n-1)!}{n!}\left(\frac{r-1}{j-1}\right)$$

A solution to the secretary problem

- Formal description of the problem:

$$\Pr(r) = \sum_{j=r}^{n} P(\,j \text{ is best AND before } r \text{ is an worst one}) =$$

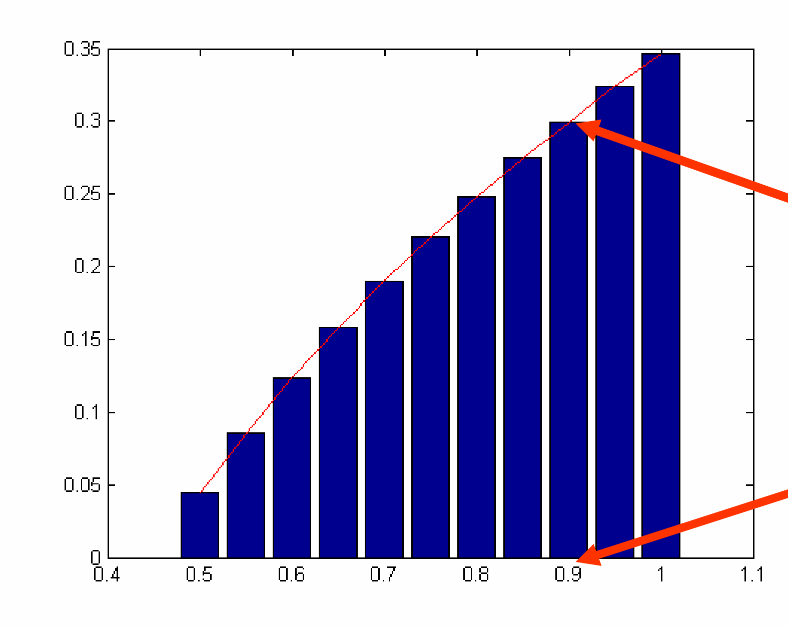
$$= \sum_{j=r}^{n} \frac{(n-1)!}{n!} \left(\frac{r-1}{j-1} \right) = \sum_{j=r}^{n} \frac{1}{n} \left(\frac{r-1}{j-1} \right)$$



A solution to the secretary problem

- An approximation if the sum is big enough

$$\Pr(r) = \sum_{j=r}^{n} \frac{1}{n}\left(\frac{r-1}{j-1}\right) = \frac{r-1}{n}\sum_{j=r}^{n}\frac{n}{j-1}\left(\frac{1}{n}\right)$$



$$\frac{n}{j-1} \quad \left(\frac{1}{n}\right)$$

Area of the rectangle

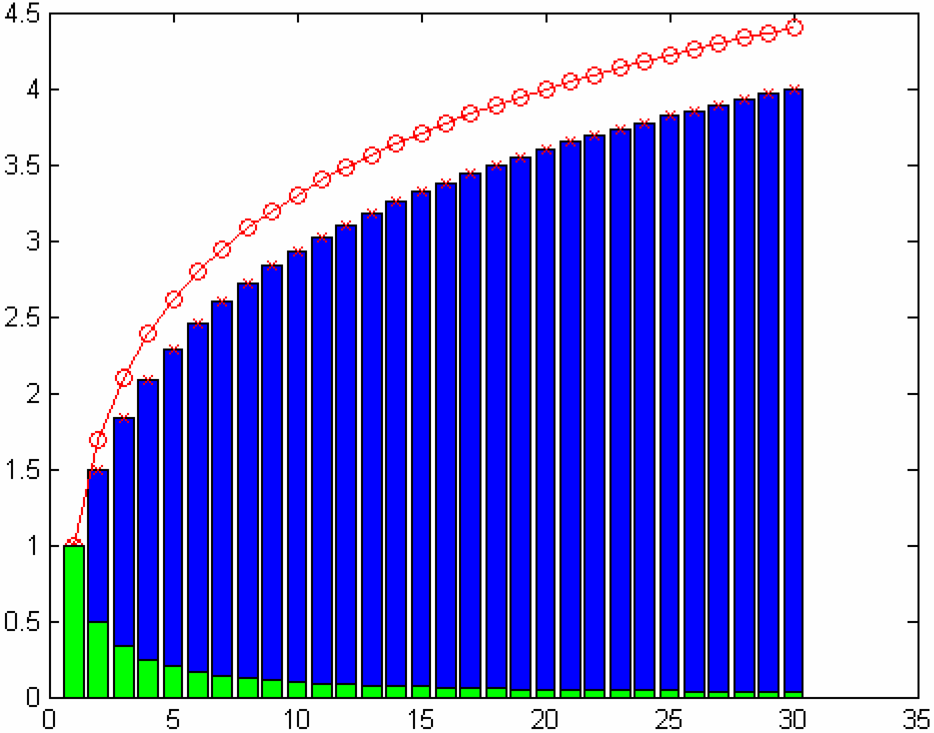
A solution to the secretary problem

- An approximation if the sum is big enough
  - We substitute the sum by the integral

$$\Pr(r) = \sum_{j=r}^{n} \frac{1}{n}\left(\frac{r-1}{j-1}\right) = \frac{r-1}{n}\sum_{j=r}^{n}\frac{n}{j-1}\left(\frac{1}{n}\right) \cong$$

$$= \left\{\begin{array}{c} C.V. \\ x = \dfrac{j-1}{n} \\ dx = \dfrac{1}{n} \end{array}\right\} = \frac{r-1}{n}\int_{\frac{r-1}{n}}^{\frac{n-1}{n}} \frac{1}{x}\,dx = \frac{r-1}{n}\left(\ln(\frac{n-1}{n}) - \ln(\frac{r-1}{n})\right)$$

Appendix. Integral of a logarithm



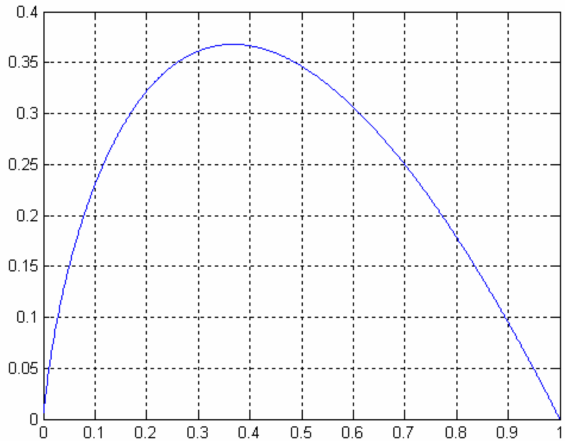
$$\int_a^b \frac{1}{x}dx = \log(b) - \log(a)$$

A solution to the secretary problem

- An approximation if the sum is big enough
  - We substitute the sum by the integral

$$\Pr(r) = \sum_{j=r}^{n} \frac{1}{n}\left(\frac{r-1}{j-1}\right) = \frac{r-1}{n}\left(\ln(\frac{n-1}{n}) - \ln(\frac{r-1}{n})\right) \cong -\frac{r-1}{n}\ln(\frac{r-1}{n})$$

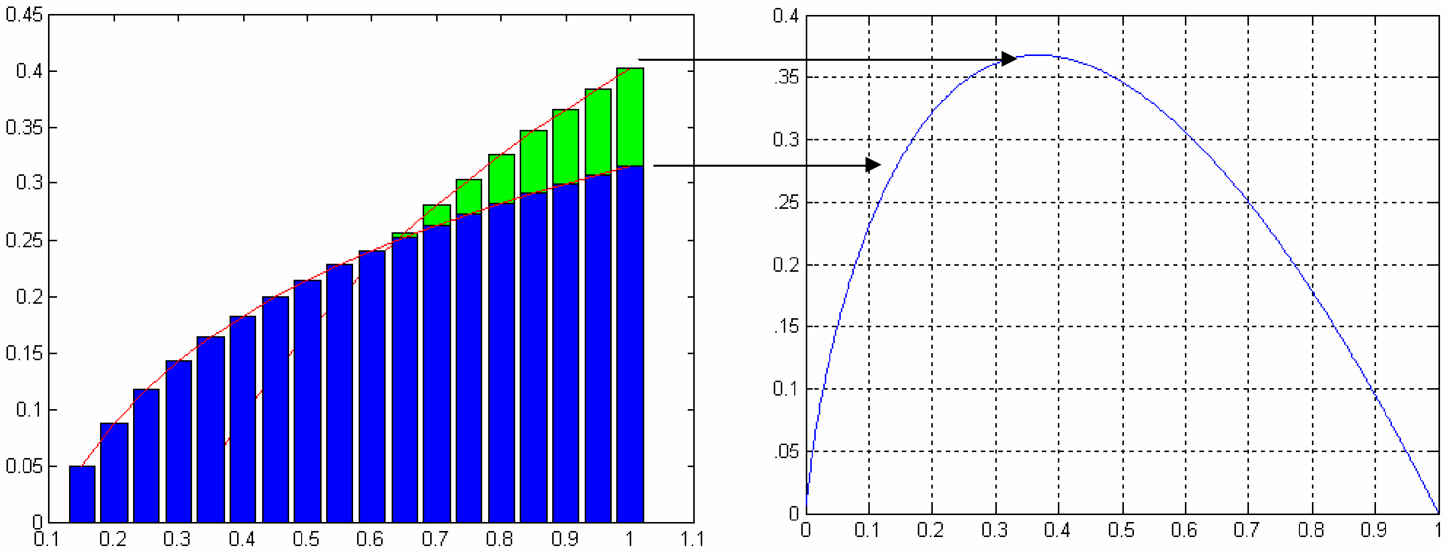
$$\Pr(r) \cong -x\log(x)$$



A solution to the secretary problem

- How good is the approximation ?
  - N=50, r=37% and r=15%

$$\Pr(r) = \sum_{i=r}^{n} \frac{1}{n}\left(\frac{r-1}{j-1}\right) \cong -\frac{r-1}{n}\ln\left(\frac{r-1}{n}\right)$$



Urn Problem

- ***n*** balls are randomly distributed into ***n*** urns. Compute the probability that:
  1. A given urn is empty
  2. At least one urn is empty
  3. A given urn is the only empty
  4. There is only one empty urn

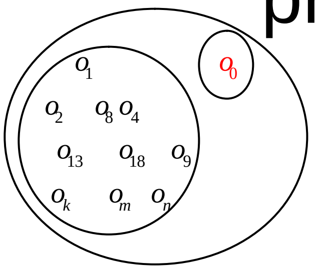
Urn Problem

- A given urn is empty
  - The balls can be distributed into (n-1) urns leaving a given one empty.

$$\Pr(A \quad Given \quad Urn) = \frac{(n-1)^n}{n^n}$$

Urn Problem

- At least one urn is empty
  - We should compute the complementary probability



$$\Pr\left(\Omega - \{o_0\}\right) = \Pr\left(o_1\right) + \Pr\left(o_2\right) + \Pr\left(o_3\right) + \cdots = 1 - \Pr\left(o_0\right)$$

  - There are n! ways of asigning one ball to one urn

$$\Pr(At \quad least \quad one \quad Urn) = 1 - \frac{n!}{n^n}$$

Urn Problem

- A given urn is the only empty
  - All the others except one have only one ball.
  - Ways of selecting a couple of balls $\binom{n}{2}$
  - Ways of distributing the balls: (n-1)!
    - Note that if the two balls are substituted by a new ball with different colour we have an equivalent problem.

$$\Pr = \binom{n}{2}\frac{(n-1)!}{n^n} = \frac{n!(n-1)}{2n^n}$$

Urn Problem

- There is only one empty urn
  - It is the probability that the empty urn is either the first or the second or etc.

$$\Pr(u_1)+\Pr(u_2)+\Pr(u_3)+\cdots+\Pr(u_n)$$

$$\Pr = n\binom{n}{2}\frac{(n-1)!}{n^n} = \binom{n}{2}\frac{(n)!}{n^n}$$

An Election Problem (Markov)

- In an election with two candidates
  - A gets n votes
  - B gets m votes
- Compute the probability of the event:
  - E={A is always ahead in the count of votes}