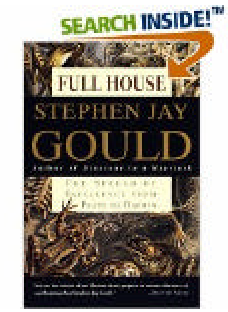
A first approach to probability

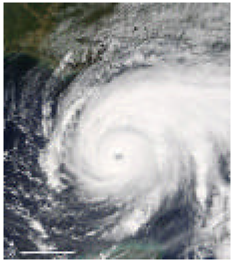
- Necessary and Contingence:
  - Cannot be other way: logic/axiomatical systems
  - Can be different: Reality
- P.S. Laplace:
  - Celestial Mechanics/Theorie analitique des probabilitees.
- Objective: Laws of randomness
  - Necesity in the contingence ☺

Full House: The Spread of Excellence from Plato to Darwin by Stephen Jay Gould

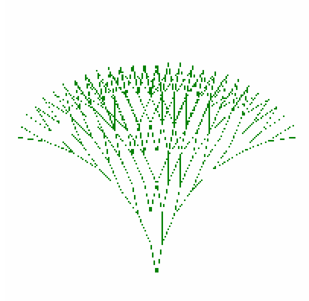


Meanings of Probability

- Relative frequency
- Beliefs:
  - Once in a life
  - Odds in gambling
  - Prediction



- Axiomatic system.
  - Subset of measure theory



Meanings of Probability

- Related words:
  - Probability
  - Likelihood
  - Chance
  - Prospect
  - Odds
  - Possibility

- Ethymologies
  - **Probable** [14th century. Directly or via French from Latin probabilis "provable, plausible," from probare (see prove).]Microsoft® Encarta®
  - **Random** [Mid-17th century. From Old French randon "impetuosity, rush" (the original sense in English), from randir "to run." Ultimately from a prehistoric Germanic base (probably also the ancestor of English run).]
  - **Chance** [13th century. Via Anglo-Norman from, ultimately, late Latin cadentia "falling," from the present participle of Latin cadere "to fall."]

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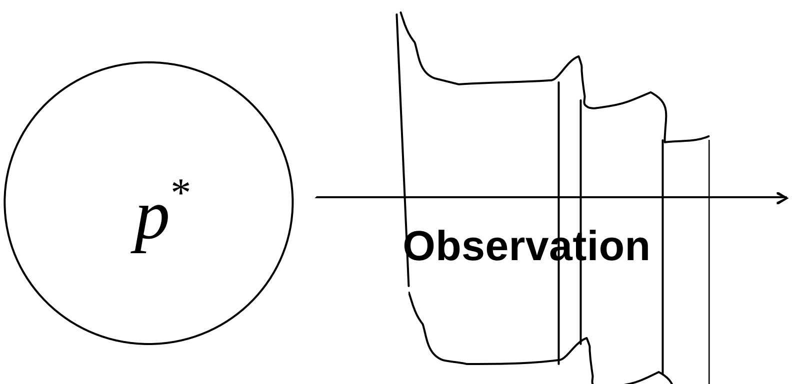
Meanings of Probability

- Relative frequencies:
  - From the structure of the problem:
    - Die, coin, urns, etc.

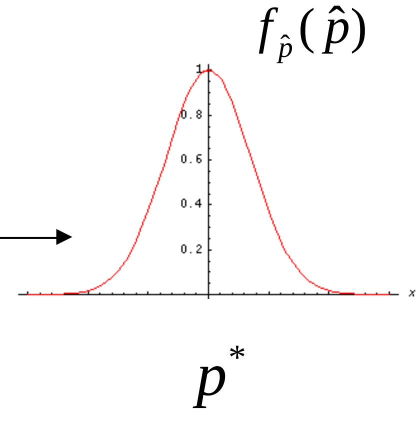
$$\Pr = \frac{Count\ of\ \ ways\ \ for\ a\ result}{Count\ of\ \ all\ \ possible\ results}$$

  - Empirical Measures:

$$\Pr = \frac{Number\ of\ \ times\ a\ result\ appears}{Count\ of\ \ the\ number\ of\ \ trials}$$



$$\hat{p} = \frac{favorable\ cases}{possible\ cases}$$



Meanings of Probability

- Relative frequencies:
  - Examples:
    - From the structure of the problem:

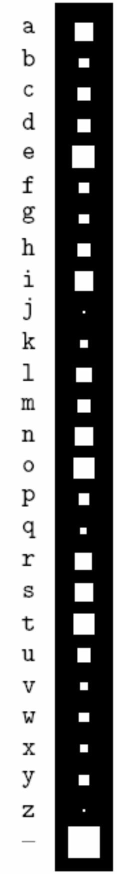
$$\Pr(a) = \frac{Count\ of\ ways\ for\ a\ result}{Count\ of\ all\ possible\ results} = \frac{1}{24}$$

    - Empirical Measures:

$$\Pr(a) = \frac{Number\ of\ times\ a\ result\ appears}{Count\ of\ the\ number\ of\ trials} = \frac{1}{17}$$

  - Which is the good one?
    - <u>*Principle of insuficient reason*</u>

i	a_i	p_i
1	a	0.0575
2	b	0.0128
3	c	0.0263
4	d	0.0285
5	e	0.0913
6	f	0.0173
7	g	0.0133
8	h	0.0313
9	i	0.0599
10	j	0.0006
11	k	0.0084
12	l	0.0335
13	m	0.0235
14	n	0.0596
15	o	0.0689
16	p	0.0192
17	q	0.0008
18	r	0.0508
19	s	0.0567
20	t	0.0706
21	u	0.0334
22	v	0.0069
23	w	0.0119
24	x	0.0073
25	y	0.0164
26	z	0.0007
27	–	0.1928

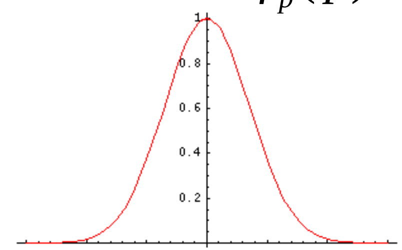


http://www.inference.phy.cam.ac.uk/mackay/itila/book.html

Kinds of probability

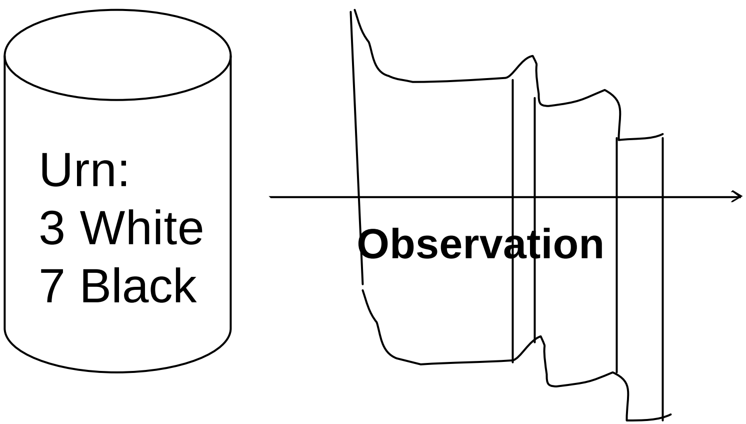
- Probability of an observation
- Probability of the cause of the observation
- Probability of the estimate of the probability.

$$f_{\hat{p}}(\hat{p})$$



$$p^*$$

$$p^* = P(white\ observation / composition\ of\ the\ urn)$$



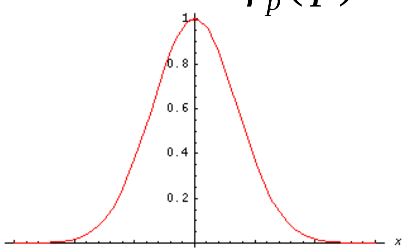
$$P(composition\ of\ the\ urn / white\ observation)$$

One of the objectives of the subject.

Memoir on the Probability of the Causes of Events
Pierre Simon Laplace
Statistical Science > Vol. 1, No. 3 (Aug., 1986), pp. 364-378
Stable URL: http://links.jstor.org/

Thomas Bayes's Essay Towards Solving a Problem in the Doctrine of Chances
G. A. Barnard; Thomas Bayes
Biometrika > Vol. 45, No. 3/4 (Dec., 1958), pp. 293-315
Stable URL: http://links.jstor.org/

$f_{\hat{p}}(\hat{p})$



p^*

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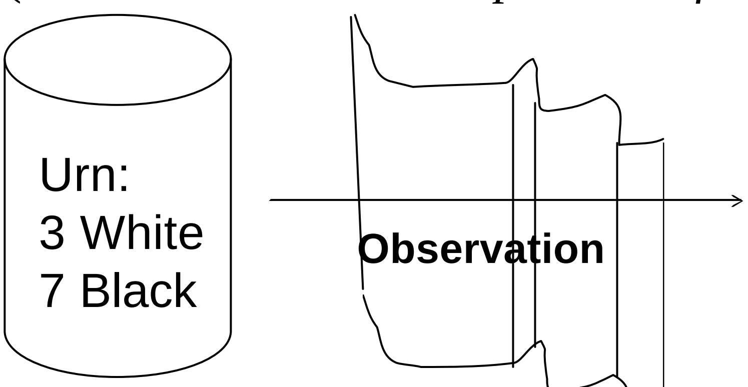
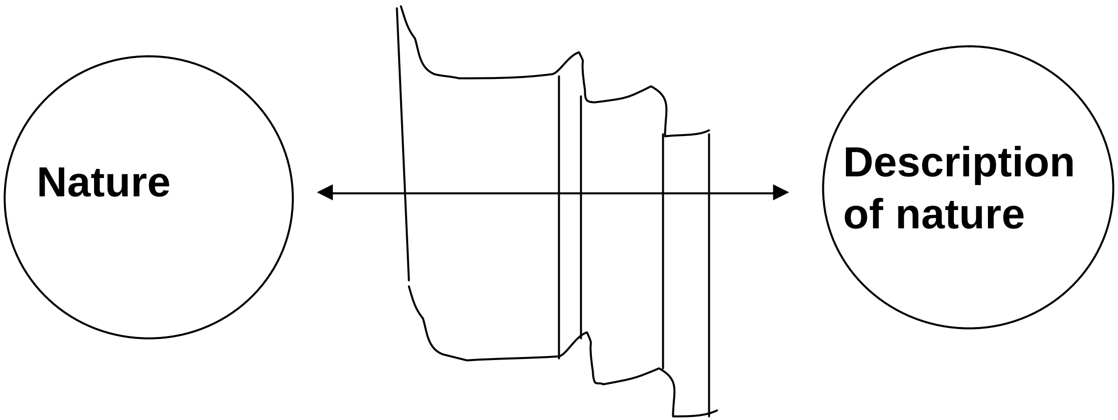


$$P(composition\ of\ the\ urn / white\ observation)$$

Meanings of ‘model’

- Engineering vs. Mathematics/logic



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http://humanum.arts.cuhk.edu.hk/humftp/Fine_Arts/Gallery/matisse/matisse.jpg

D’Alembert’s mistake

- Entry in ‘L’Encyclopedie’: In Croix ou Pile
  - d'Alembert introduced his famous error that the probability that at least one head should appear in two consecutive tosses of a fair coin is 2/3 rather than 3/4. In addition,
- Problem: how much are the odds that one will bring heads in playing two successive tosses.

http://www.cs.xu.edu/math/Sources/Dalembert/index.html

Definition of odds

- Odds in gambling
  - Is used to compare the unfavorable with the favorable possibilities

$$1:1 \rightarrow \frac{1}{1+1}$$

$$2:1 \rightarrow \frac{1}{2+1}$$

$$a:b \rightarrow \frac{b}{a+b}$$

D’Alembert’s mistake

- Reasoning:
  - For in order to take here only the case of two tosses, is it not necessary to reduce to one the two combinations which give heads on the first toss? For as soon as heads comes one time, the game is finished, & the second toss counts for nothing. So there are properly only three possible combinations:
  - Therefore the odds are 2 against 1

***Heads*, first toss.**
***Tails, heads*, first & second toss.**
***Tails, tails*, first & second toss.**

http://www.cs.xu.edu/math/Sources/Dalembert/index.html

D'Alembert's mistake Solution

- Reasoning:

$$\Pr = \frac{Count\ of\ ways\ for\ a\ result}{Count\ of\ all\ possible\ results}$$

$$\Pr = \frac{3}{4}$$

$$\Pr = \frac{2}{3}$$

Green Coin.	**Blue Coin.**
Heads.	***Heads.***
***Tails*.**	***Heads.***
***Heads*.**	***Tails.***
***Tails*.**	***Tails.***

***Heads*, first toss.** ***Heads*, Second toss.**	**Mistake**
***Tails, heads*, first & second toss.**	
***Tails, tails*, first & second toss.**	

How to avoid the Mistake

- When having objects of the same kind, always number or colour in order to distinguish them. **Mistake**

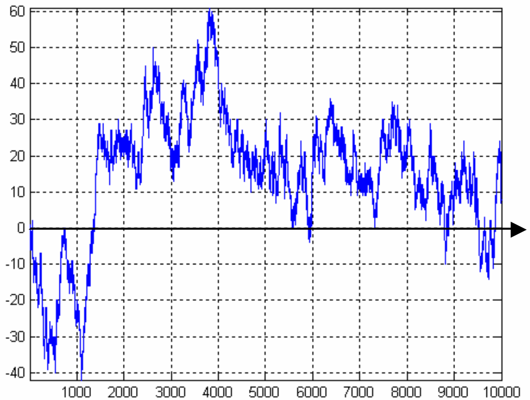
***Green Coin*.**	***Blue Coin*.**
Heads.	***Heads.***
***Tails*.**	***Heads.***
***Heads*.**	***Tails.***
***Tails*.**	***Tails.***

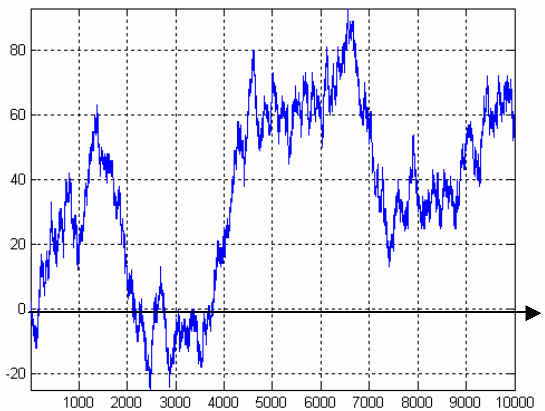
***Heads*, first toss.** ***Heads*, Second toss.**
***Tails, heads*, first & second toss.**
***Tails, tails*, first & second toss.**

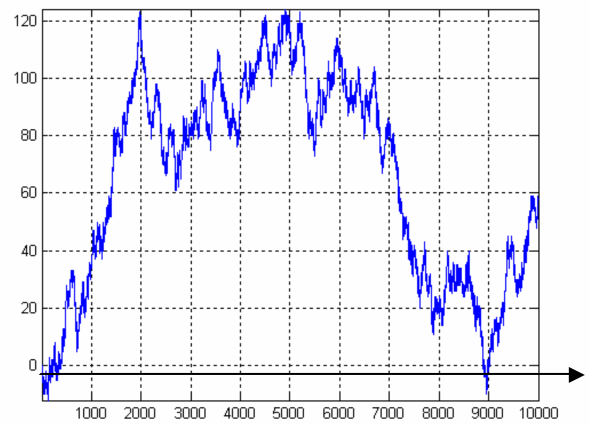
- Problem: A boy opens the door and you know that the familiy has two children, which is the probability that the boy has a sister?

Mistakes of Intuition

- Daniel and Nicolas flip a coin. If face then Daniel receives a Florin, otherwise pays it.
- Temporal evolution (10000 flips):

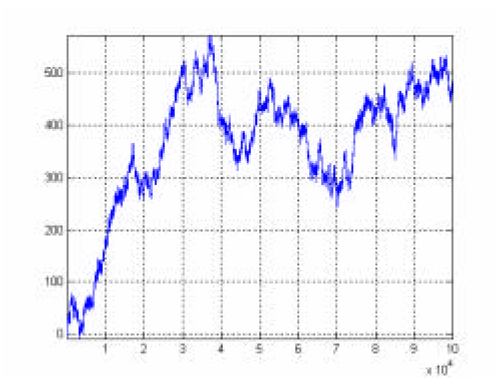


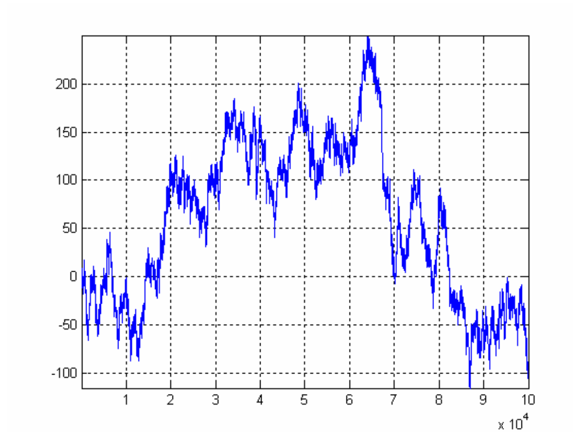


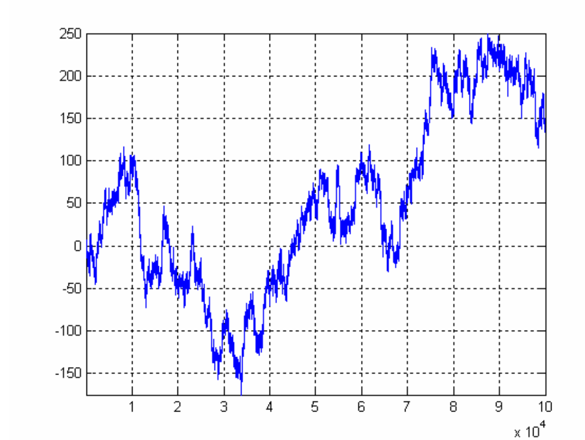


Mistakes of Intuition

- What happens in a very long game?
  - 1000000 flips



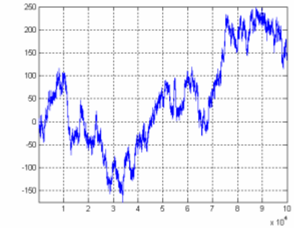


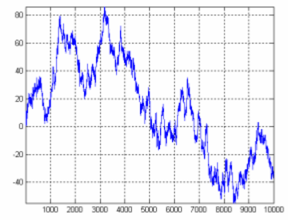


- Look the same!

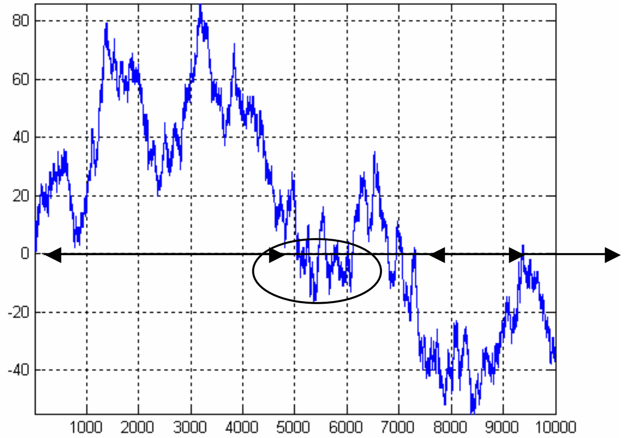
Mistakes of Intuition

- Take into account:
  - Shape independent of the scale





  - Most of the time one winner
  - Zero crossings clustered
  - Zero crossings get sparse
  - Central limit theorem
    - (Gausian?)



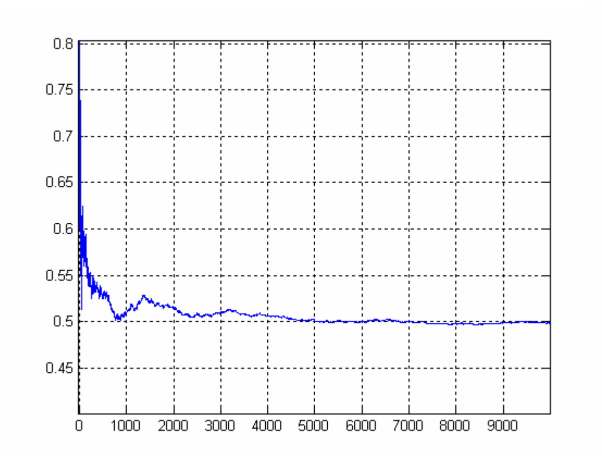
Mistakes of intuition

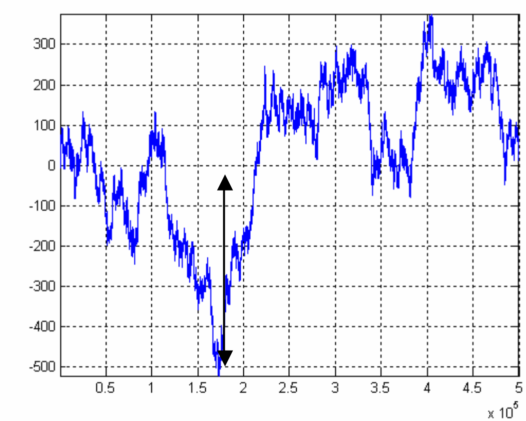
Intuition corresponds to ratio.

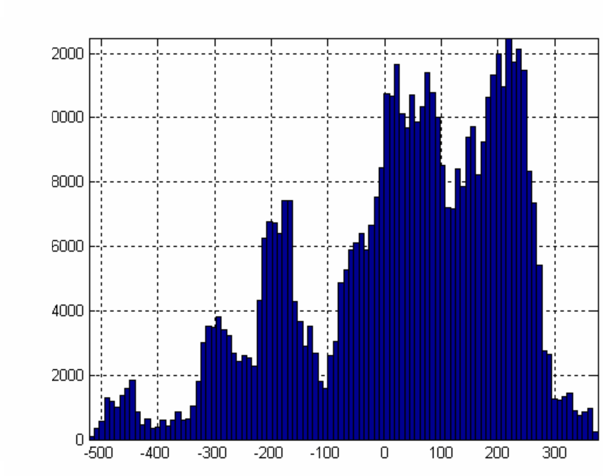
- Convergence on ratio.
- Difference gets as bigger !

$$\frac{favorable}{(favorable+unfavorable)} \rightarrow \frac{1}{2}$$

$$\left|favorable-unfavorable\right| \rightarrow \infty$$







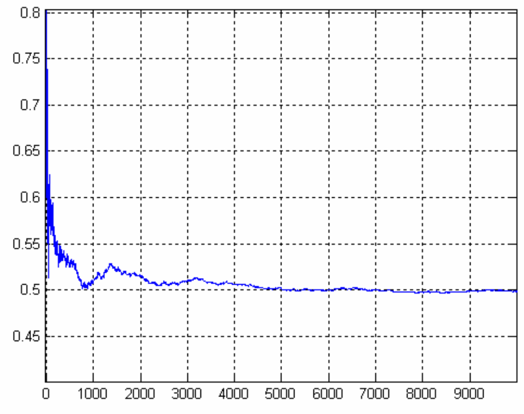
Mistakes of intuition

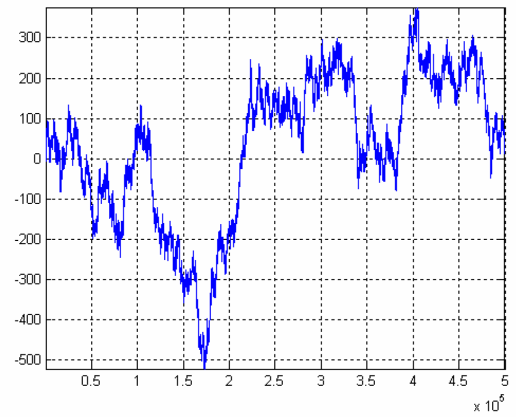
Mathematical Correspondences

$$\Pr(H) = \lim_{N\to\infty} \frac{N_H}{N}$$

$$\left|\Pr(H) - \frac{N_H}{N}\right| < \epsilon$$

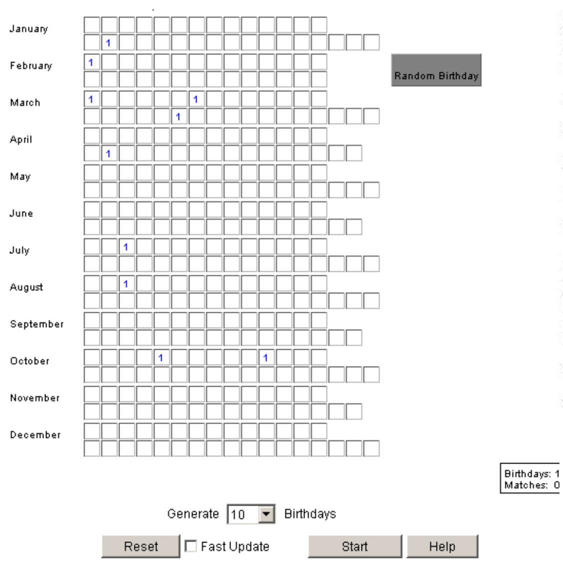
$$\frac{favorable}{(favorable + unfavorable)} \to \frac{1}{2} \qquad |favorable - unfavorable| \to \infty$$

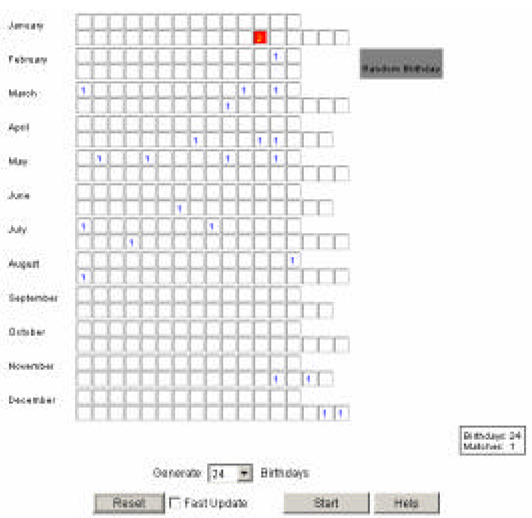


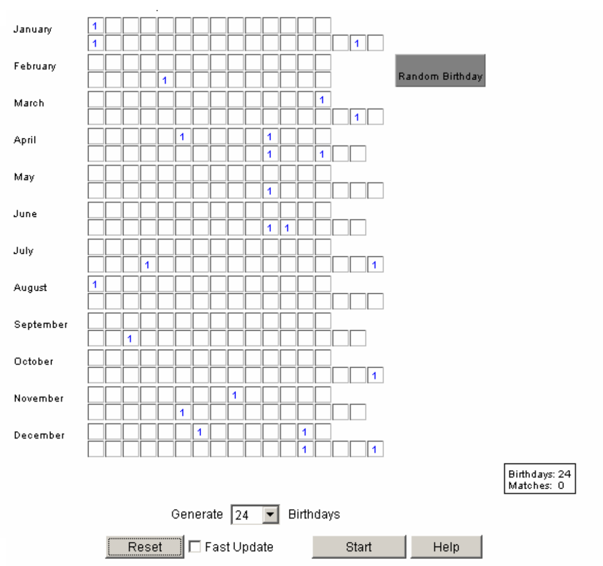


Mistakes of intuition

- Birthday problem/ Coincidences
  - 10 & 24 birthdays

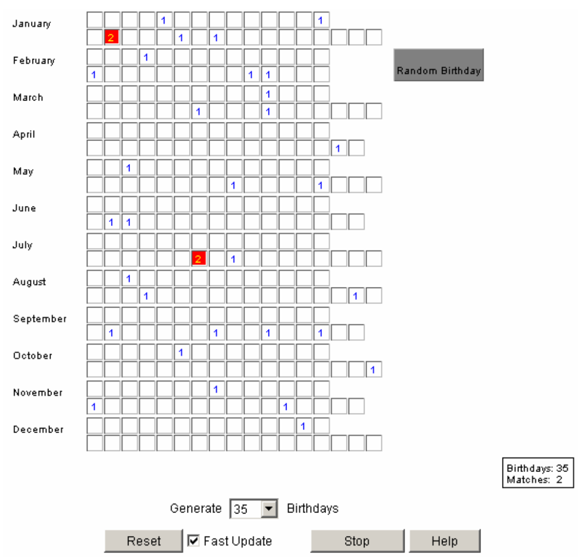


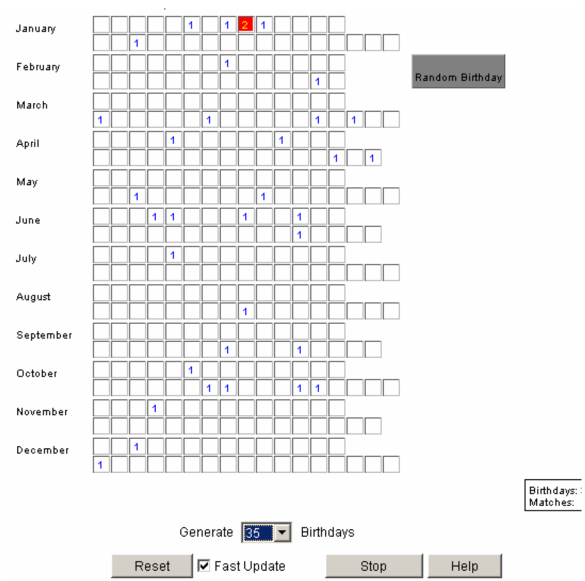


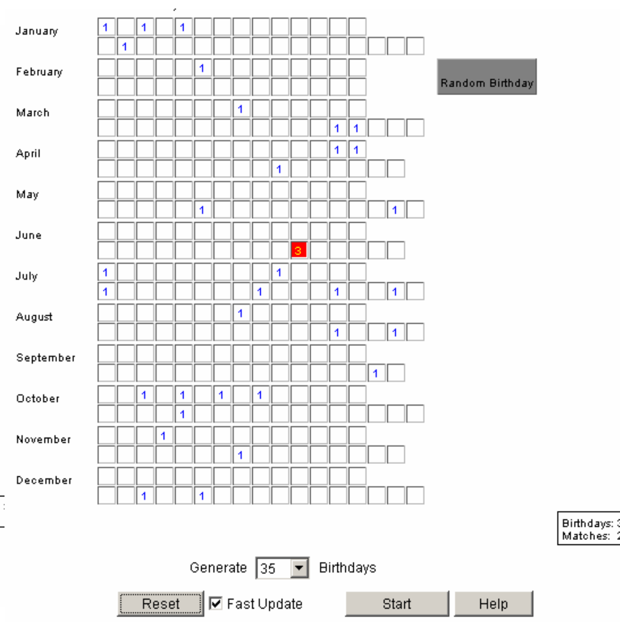


Mistakes of intuition

- Birthday problem / Coincidences
  - 35 birthdays
  - Three conscutive simultations yield days with two and three birthdays.

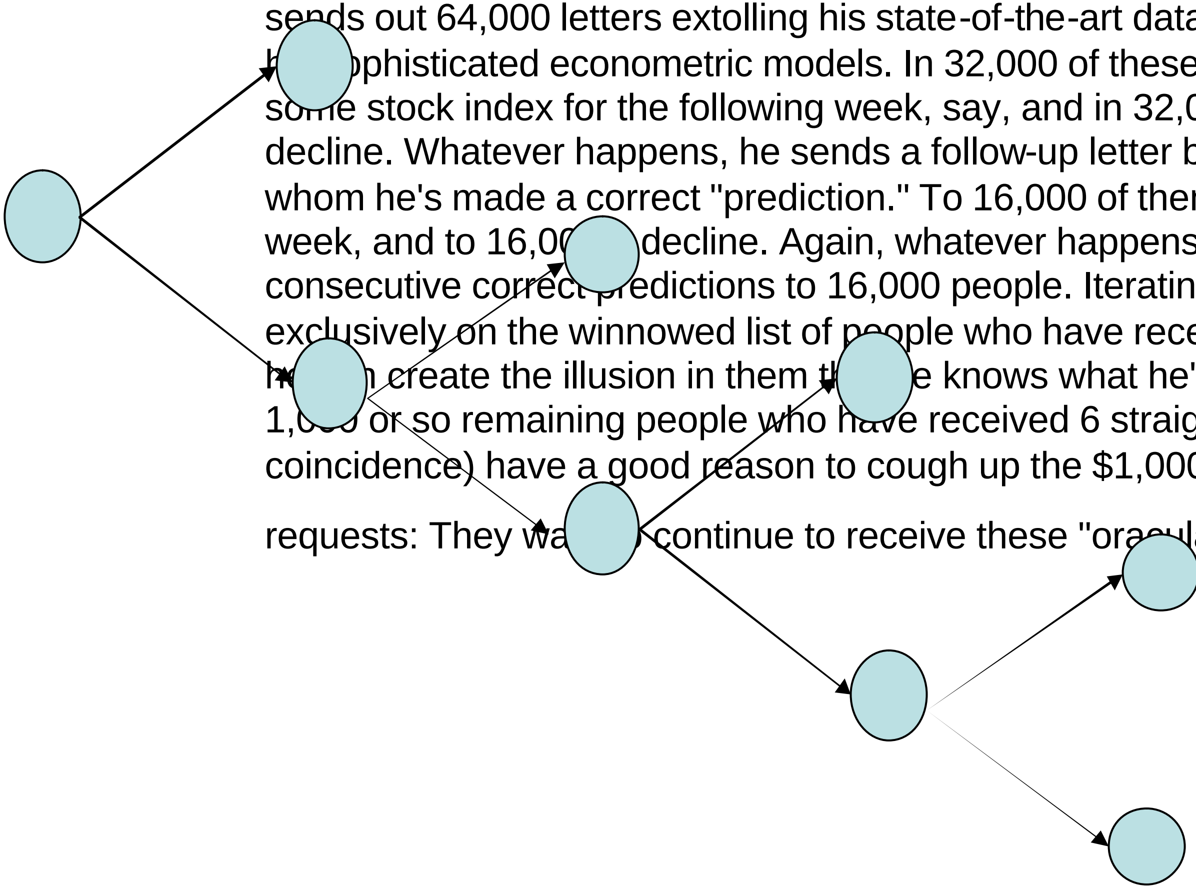






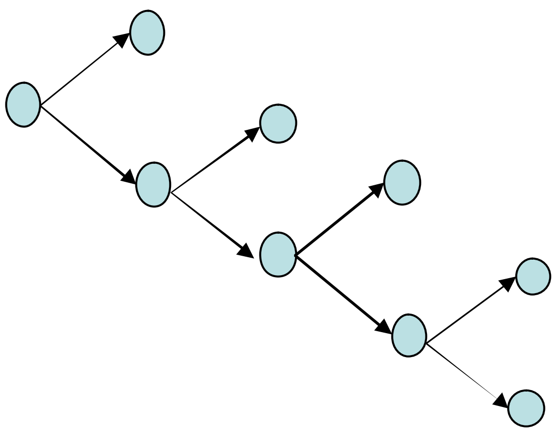
Origin of the Black swan

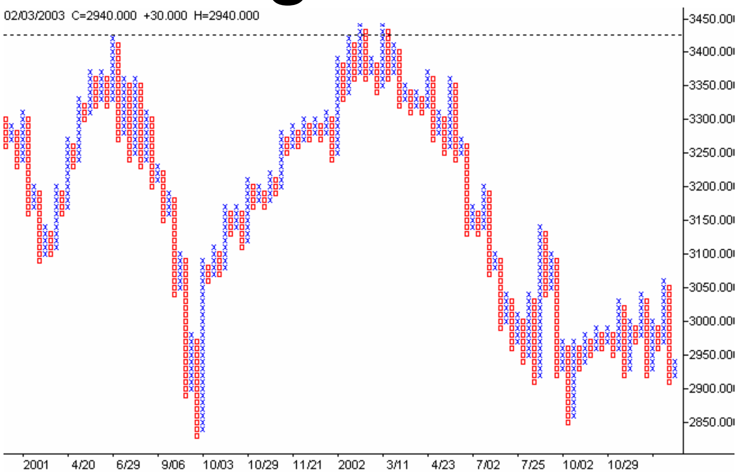
- Paulo’s piramidal game
  - One somewhat different example concerns the publisher of a stock newsletter who sends out 64,000 letters extolling his state-of-the-art database, his inside contacts, and his sophisticated econometric models. In 32,000 of these letters he predicts a rise in some stock index for the following week, say, and in 32,000 of them he predicts a decline. Whatever happens, he sends a follow-up letter but only to those 32,000 to whom he's made a correct "prediction." To 16,000 of them he predicts a rise for the next week, and to 16,000 a decline. Again, whatever happens, he will have sent 2 consecutive correct predictions to 16,000 people. Iterating this procedure of focusing exclusively on the winnowed list of people who have received only correct predictions, he can create the illusion in them that he knows what he's talking about. After all, the 1,000 or so remaining people who have received 6 straight correct predictions (by coincidence) have a good reason to cough up the $1,000 the newsletter, publisher requests: They want to continue to receive these "oracular" pronouncements.

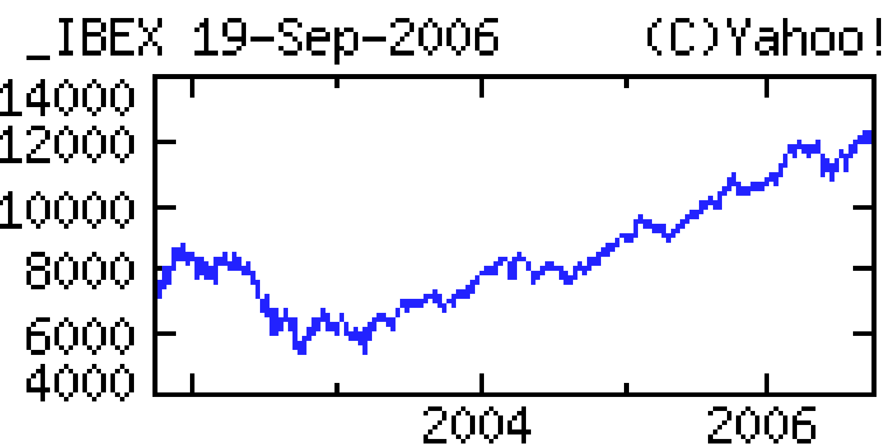


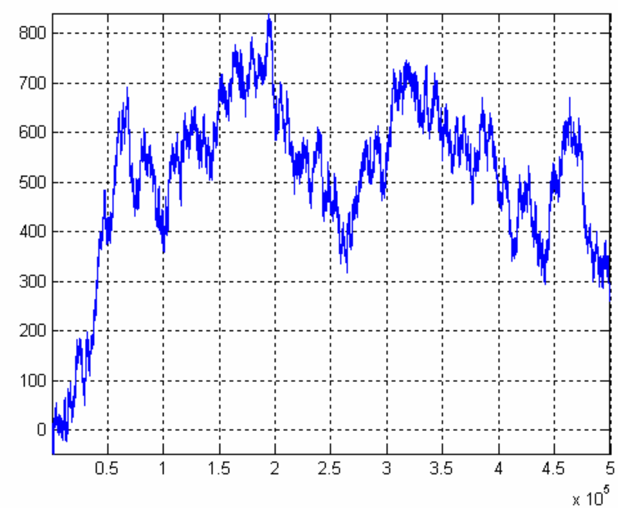
Origin of the Black swan

- Stock/gold index as a random game



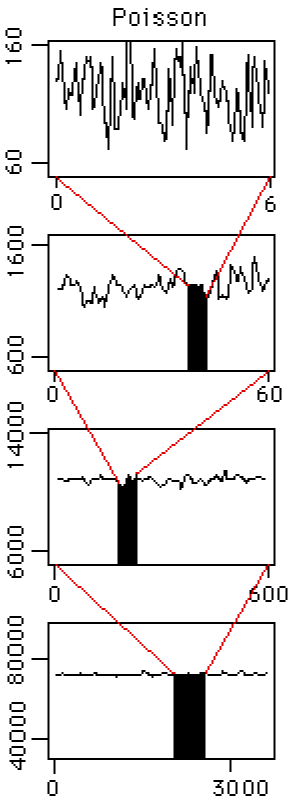






Internet Traffic

- Sudden peaks

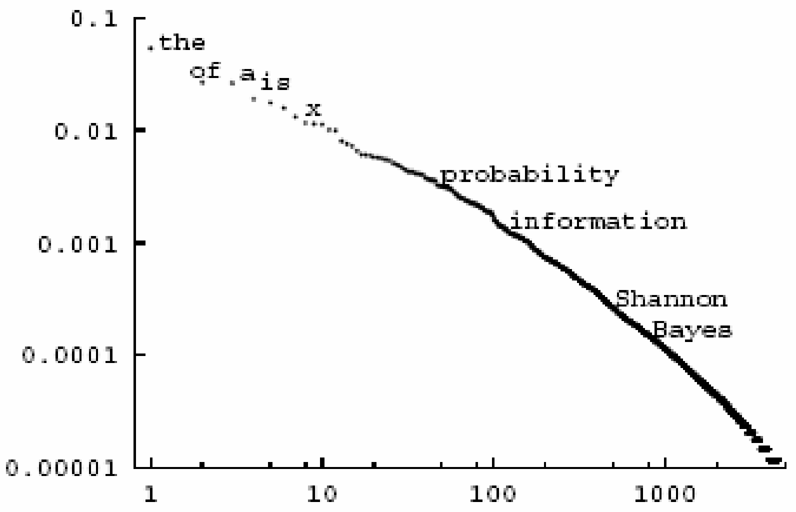


llinger, W., and Paxson, V., "Where mathematics meets the internet,"
otices of the AMS 45 (1998), 961-970.

http://classes.yale.edu/fractals/Panorama/ManuFractals/Internet/Internet4.htm

Word frequencies

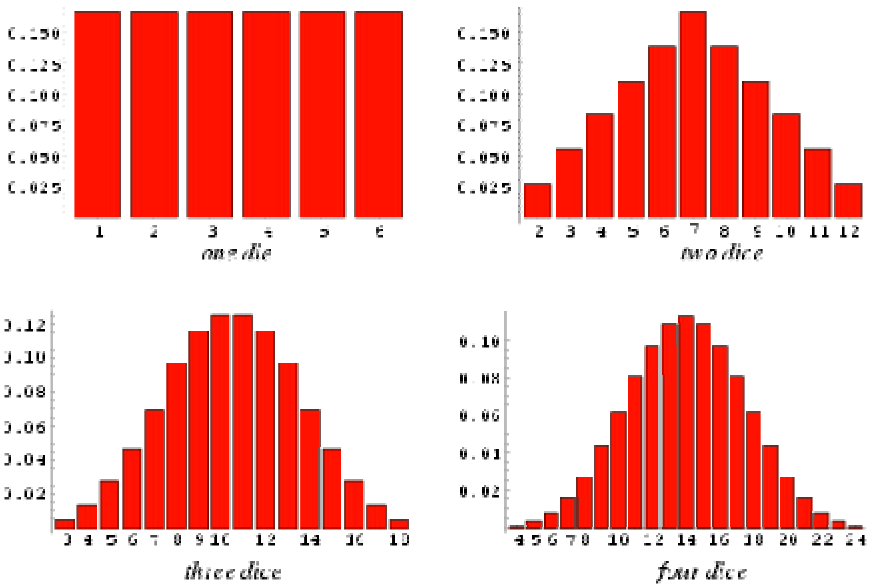
- Sudden peaks

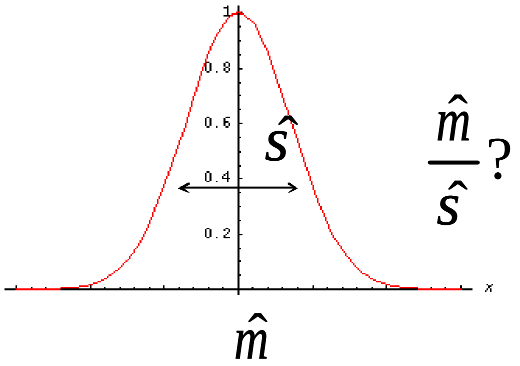


http://www.inference.phy.cam.ac.uk/mackay/itila/book.html

Sum of random variables

- Example: sum of points of *n* dice





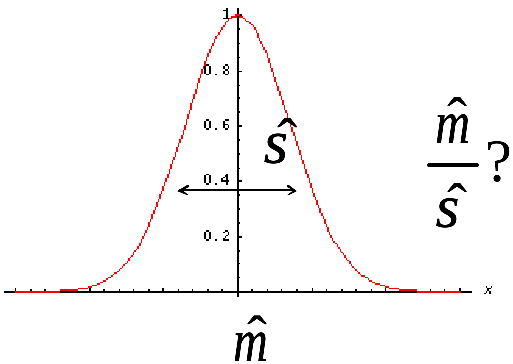
Law of large numbers

Information Theory

http://mathworld.wolfram.com/Dice.html

Sum of random variables

- What happens when n -> 00



$$\frac{\hat{m}}{\hat{s}}?$$

$$\lim_{n \to \infty} P\left(|X - \mu| \geq \epsilon\right) = 0.$$

$$P\left(|X - \mu| \geq \epsilon\right) \leq \frac{\operatorname{var}(X)}{\epsilon^2} = \frac{\sigma^2}{n\,\epsilon^2}.$$

http://mathworld.wolfram.com/WeakLawofLargeNumbers.html

Information Theory

- Underlying idea:An example with words

$$\Omega = \{a, b, c, \cdots z\}$$

$$P(\{the\}), P(\{an\}), P(\{house\}), P(\{boy\})$$

$$P(\{thehouseoftheboy\}), \cdots, P(\{maryhadalittlelamb\})$$

$$P(\{thehouseoftheboy \cdots maryhadalittlelamb\}) = \cdots = P(\{TEXT\ OF\ A\ BOOK\}) \rightarrow 1/2^M$$

$$P(\{TRYFGHLÑPMSWZ \cdots ZZDDRVBJKP\}) = \cdots = P(\{AEEIOUO \cdots OUUAEE\}) \rightarrow 0$$

$$P(\,function\{ARBITRARY\ TEXT\}) = \begin{cases} 1 \\ ? \\ 0 \end{cases}$$

Information Theory

- Underlying idea:An example with words

$$book1 \rightarrow 0000000001$$
$$book2 \rightarrow 0000000002$$
$$\vdots$$
$$bookN \rightarrow 9999999999$$

$$P(\{thehouseoftheboy \cdots maryhadalittlelamb\}) = \cdots = P(\{TEXT\ OF\ \ A\ BOOK\}) \rightarrow 1/2^M$$
$$P(\{TRYFGHLÑPMSWZ \cdots ZZDDRVBJKP\}) = \cdots = P(\{AEEIOUO \cdots OUUAEE\}) \rightarrow 0$$

Information Theory

- Special Branch of prob. theory.
  - Underlying idea: Given a Bernouilli trial

$$\Omega = \{H, F\}$$

$$P(\{HF\}) = P(\{FH\}) = P(\{FF\}) = P(\{HH\}) = 1/4$$

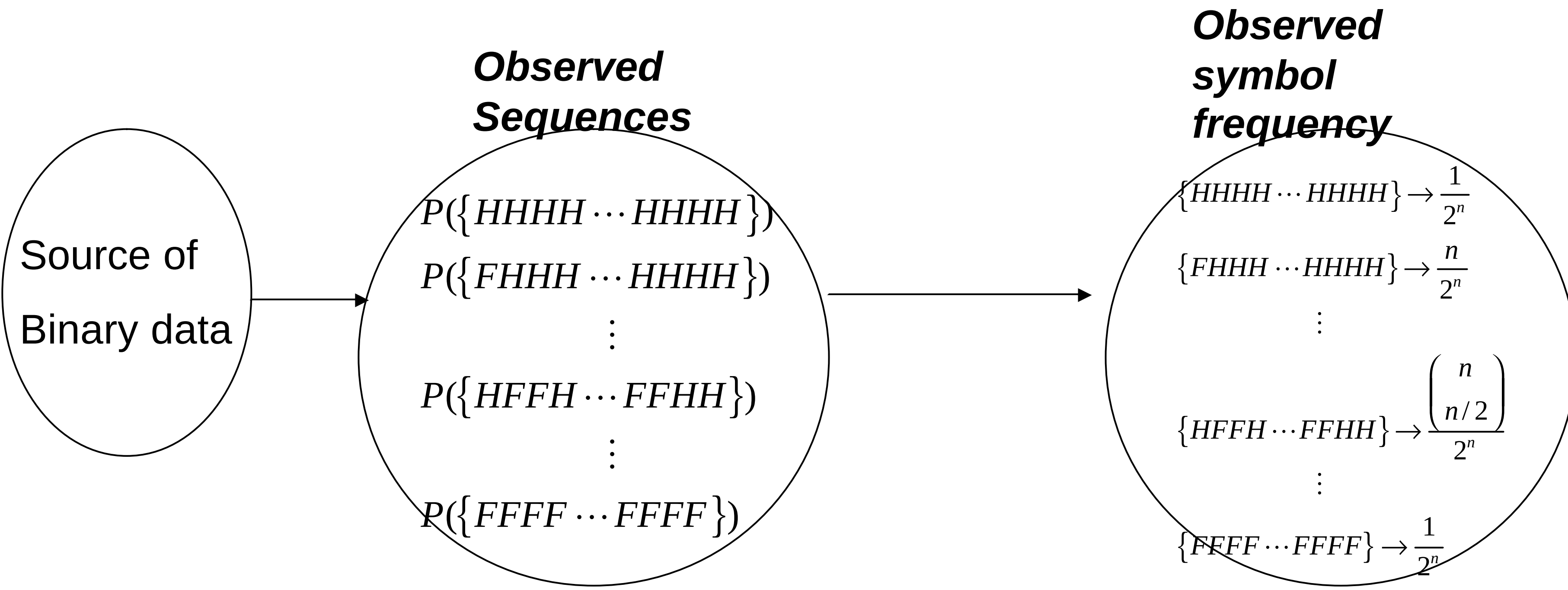
$$P(\{HFFHFF\}) = \cdots = P(\{FHFFHH\}) = 1/2^6$$

$$P(\{HFF \cdots HFF\}) = \cdots = P(\{FHF \cdots FHH\}) = 1/2^n$$

$$P(\,function\{HFFH \cdots FFHH\}) = \begin{cases} 1 \\ ? \\ 0 \end{cases}$$

Information Theory

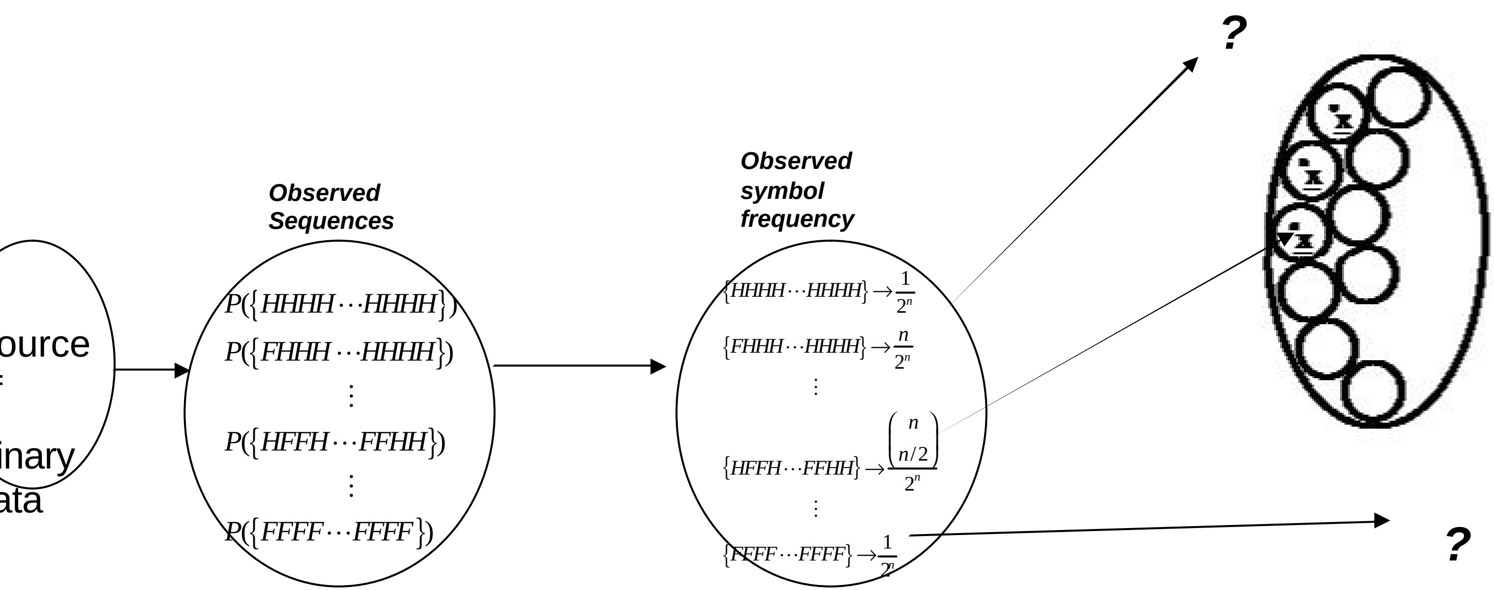
- Special Branch of prob. Theory.
  - Underlying idea: Given a Bernouilli trial



$$P(\{HFF \cdots HFF\}) = \cdots = P(\{FHF \cdots FHH\}) = 1/2^n$$

Information Theory

- Special Branch of prob. Theory.
  - Underlying idea: Given a Bernouilli trial



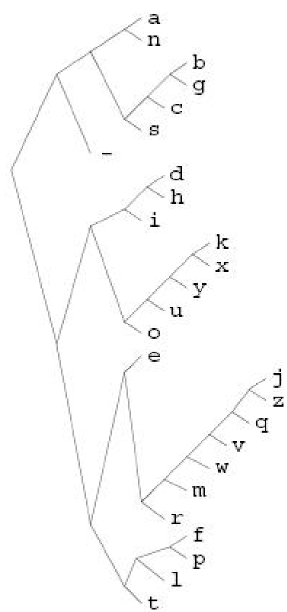
$$P(\{HFF\cdots HFF\}) = \cdots = P(\{FHF\cdots FHH\}) = 1/2^n$$

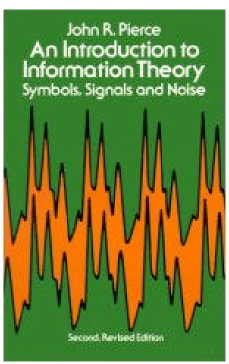
Probability of not being able to code a symbol?

Application

- Coding/criptography
  - How to use probabilities for compression?
  - Code and encript or encript and code?

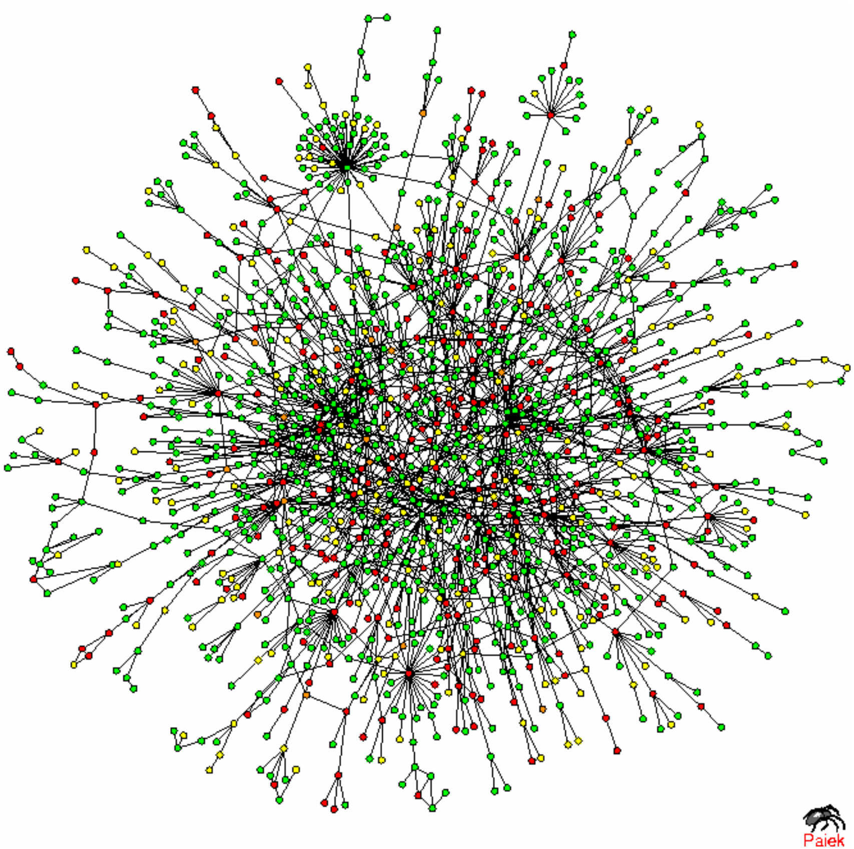
a_i	p_i	$\log_2 \frac{1}{p_i}$	l_i	$c(a_i)$
a	0.0575	4.1	4	0000
b	0.0128	6.3	6	001000
c	0.0263	5.2	5	00101
d	0.0285	5.1	5	10000
e	0.0913	3.5	4	1100
f	0.0173	5.9	6	111000
g	0.0133	6.2	6	001001
h	0.0313	5.0	5	10001
i	0.0599	4.1	4	1001
j	0.0006	10.7	10	1101000000
k	0.0084	6.9	7	1010000
l	0.0335	4.9	5	11101
m	0.0235	5.4	6	110101
n	0.0596	4.1	4	0001
o	0.0689	3.9	4	1011
p	0.0192	5.7	6	111001
q	0.0008	10.3	9	110100001
r	0.0508	4.3	5	11011
s	0.0567	4.1	4	0011
t	0.0706	3.8	4	1111
u	0.0334	4.9	5	10101
v	0.0069	7.2	8	11010001
w	0.0119	6.4	7	1101001
x	0.0073	7.1	7	1010001
y	0.0164	5.9	6	101001
z	0.0007	10.4	10	1101000001
–	0.1928	2.4	2	01





One of the objectives of the subject

- Learn tools for making the transition:



$$\rightarrow \quad P(k) \sim (k + k_0)^{-g} \exp(-\frac{k + k}{k_t}$$

Taken from:

The architecture of complexity: From the topology of the www to the cell's genetic network

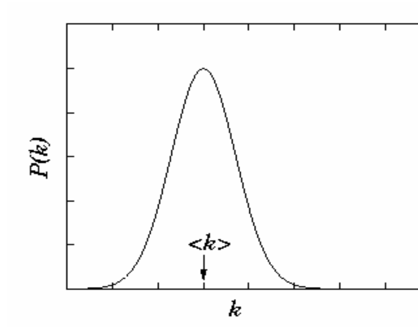
Albert-László **Barabási**

One of the objectives of the subject

- Learn how to read Prob.Density Functions :

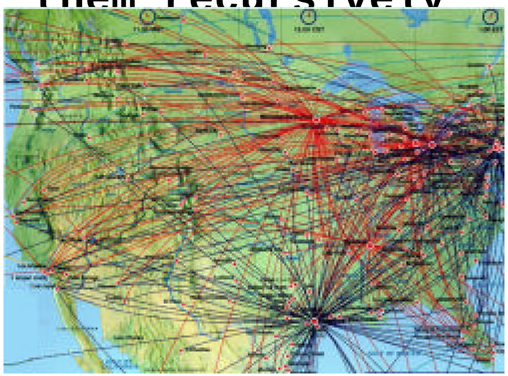
Exponential

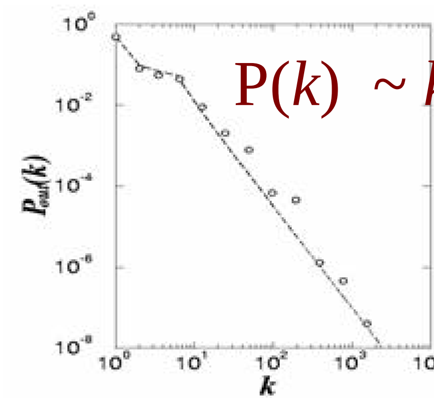
Network



Expected

Scale-free





$P(k) \sim k^{-\gamma}$

Found

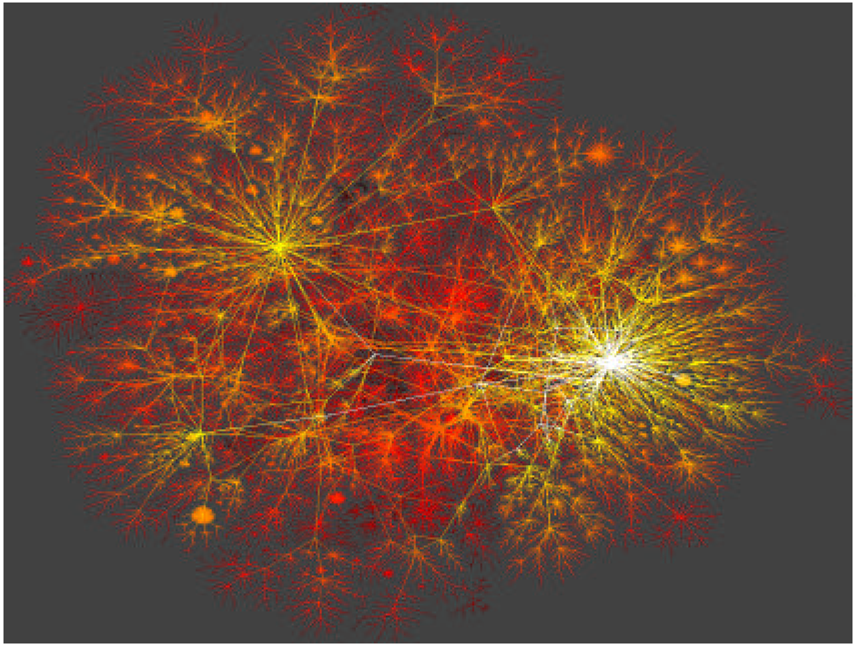
Taken from:

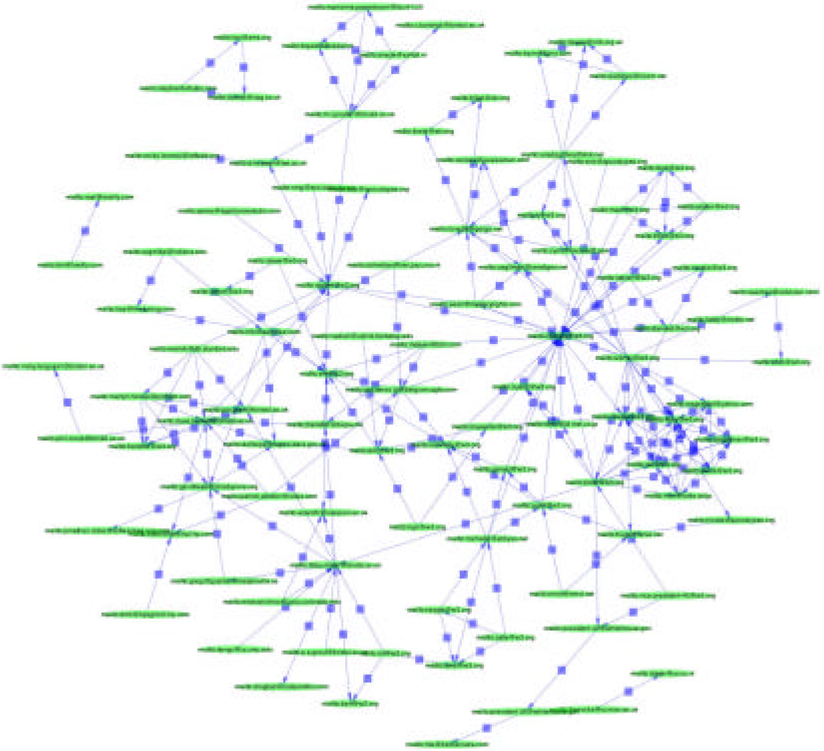
The architecture of complexity: From the topology of the www to the cell's genetic network

Albert-László **Barabási**

Similarities between natural graphs

- Semantic map vs. Physical connections in internet

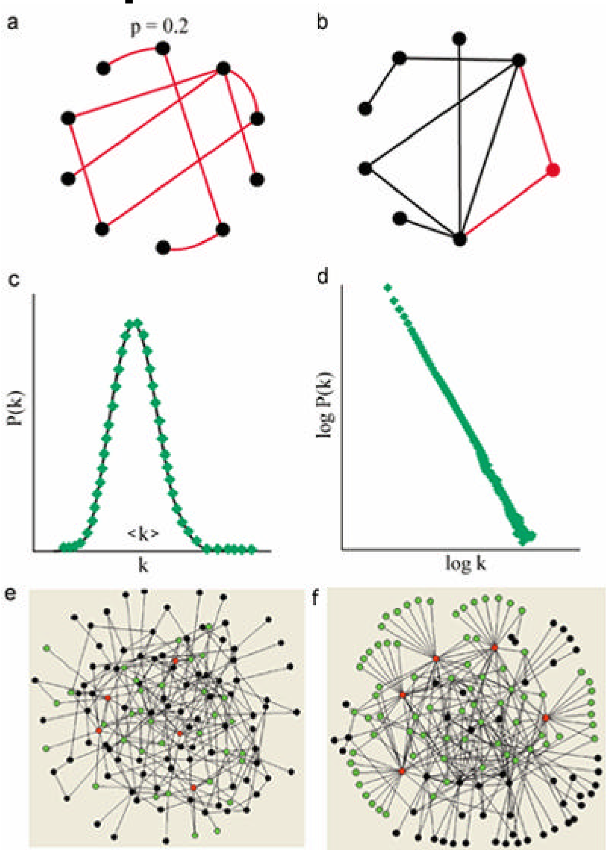




http://rdfweb.org/2002/02/foafpath/

One of the objectives of the subject

- How to construct the mathematical expression/description of the system?



Description of the creation process

$$P(k) \sim k^{-g}$$

$$(g = 3)$$

$$f(k;\lambda) = \frac{e^{-\lambda}\lambda^{k}}{k!},$$

http://physicsweb.org/articles/world/14/7/9/1#pw1407091

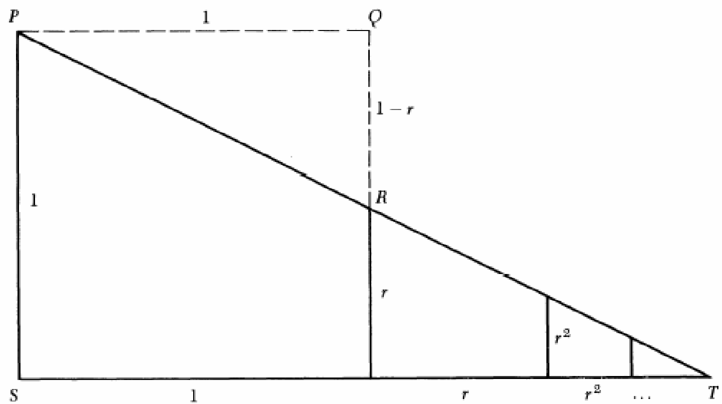
The physics of the Web

One of the objectives of the subject

- How to compute difficult probabilities?

$$\sum_{n=0}^{x} \binom{n+r-1}{r-1} p^r (1-p)^n \xrightarrow{?} 1 - \frac{(1-p)^{x+1} p^r \, \Gamma(x+r+1) \, {}_2\tilde{F}_1(1, x+r+1; x+2; 1-p)}{\Gamma(r)}$$

Proof Without Words



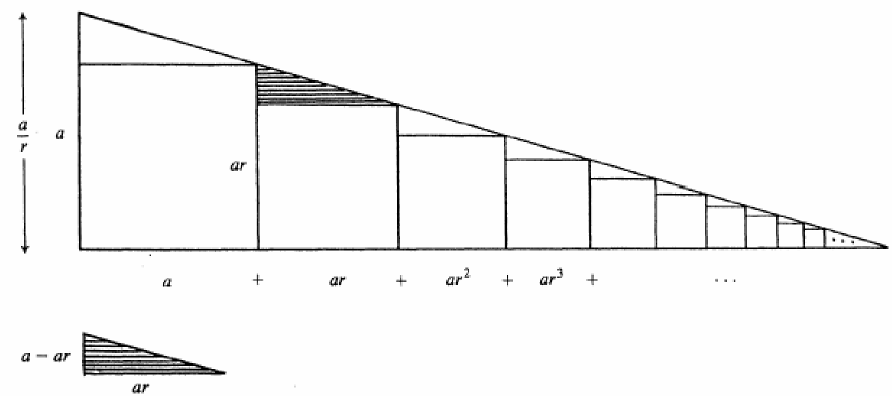
$\triangle PQR \approx \triangle TSP.$

$\therefore 1 + r + r^2 + \cdots = \dfrac{1}{1-r}.$

—Benjamin G. Kl
Irl C. Bivens
Davidson Colle
Davidson, NC 2

Proof without Words:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$



$$\frac{a-ar}{ar} = \frac{\frac{a}{r}}{a + ar + ar^2 + ar^3 + \cdots}$$

$$\Rightarrow a + ar + ar^2 + ar^3 + \cdots$$

$$= \frac{a}{1-r}.$$

J. H. Webb
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