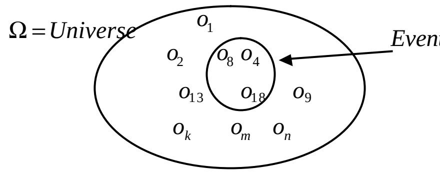
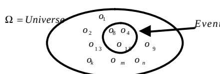
Mathematical Model of Probability

First concepts

- Sample space $\Omega = \{o_1, o_2, \cdots, o_n\}$
- Outcomes $\{o_i\}$
- Events $A = \{o_{i_1}, o_{i_2}, \cdots, o_{i_k}\}$



First concepts



- Examples:
  - Toss a coin $\Omega = \{H, F\}$
  - Toss a coin until a F

$$\Omega = \{H, HF, FH, FF, HH, HHF, HFH \cdots, H_{\ldots}HF, \cdots\}$$
$$A = \{H, HF, HHF, \cdots, H_{\ldots}HF, \cdots\}$$

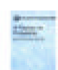
  - D'Alemberts experiment $\Omega = \{HF, FH, FF, HH\}$
    - at least one head in two tosses $A = \{HF, FH, FF\}$

Logic of Events

- Levels of concepts
  - Hierarchy of opperations

$$Logic = \{OR, AND, NOT, IMPLICATION\}$$
$$Sets = \{UNION, INTERSECTION, COMPLEMENT, INCLUSION\}$$
$$Sets = \{SUM, MULTIPLICATION, CONDITIONING\ (p(./.))\}$$

  - Travel through the hierarchy from prepositions to probabilities



A Treatise on Probability
by *John Maynard Keynes*

Logic of Events

- Scope of each concept
  - Hierarchy of opperations
  - Logic Sets Probabilities

Propositions $\rightarrow$ Relations between objects $\rightarrow$ Numbers

Valid argument $\rightarrow$ Plausible Argument

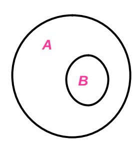
Images taken from the wikipedia

Logic of Events

Operations		
Implication $\rightarrow$	Inclusion $\rightarrow$	Condition

- Preposition
  - If it rains, I'll bring the umbrella
- Sets
  - A={It rains}, B={Bring umbrella}

    $B \subset A$
- Probabilities
  - P(B/A)=P(B)
  - P(B/Not A)= $\varnothing$



Propositions $\rightarrow$ Relations between objects $\rightarrow$ Numbers

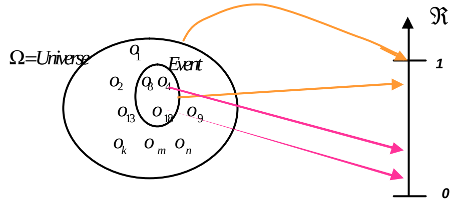
Comments

- Logic: art of thinking clearly
  - Transfer the clearness into plausible arguments.
- About Logical paradoxes and probability

The Limits of Mathematics (Discrete Mathematics and Theoretical Computer Science)
by Gregory, J. Chaitin (Hardcover - Oct 28, 2002)

The concept of probability

- Mapping $P(.): \Omega \to [0,1]$



Note: Try guessing properties of *p(.)*

The concept of probability

- Properties that should have a probability function
  - Disjoint or incompatible events

$$\text{If } B \cap A = 0 \text{ then } P(B \cap A) = P(A) + P(B)$$

  - Sure event and impossible event

$$P(\Omega) = 1 \quad P(\varnothing) = 0$$

- Def. ***Probability Space*** (Ω, P)
  - Universe of objects plus operations
  - Note similarities with an algebra.

The concept of probability

- How to construct: $P(.): \Omega \to [0,1]$
  - Coherence
  - From structure or empirical measurement?
- D'Alembert's problem (Ω, P)

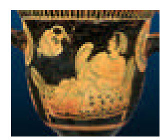
$$\Omega = \{HF, FH, FF, HH\}$$

$$P(\{HF\}) = P(\{FH\}) = P(\{FF\}) = P(\{HH\}) = 1/4$$

$$P(\{HF, FH\}) = \cdots = P(\{FF, HH\}) = 1/2$$

$$P(\{HF, FH, FF\}) = \cdots = P(\{FH, FF, HH\}) = 3/4$$

$$P(\{HF, FH, FF, HH\}) = 1$$



The concept of probability

- How to construct: $P(.): \Omega \to [0,1]$
  - Compute areas
  - Prob. a special case of *Measure theory*.

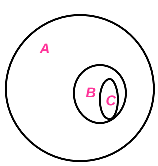
$$P(\Omega) = TotalArea = 1$$

$$P(A) = Areaof\ A$$

$$P(C) = AreaofC$$

$$P(C/B) = \frac{AreaofC}{Areaof\ B}$$

$$P(B/C) = Areaof\ C$$



Intersection

- How to deal with the intersection
  - Example: Die $\Omega = \{1,2,3,4,5,6\}$
    - Set of events

$$A = \{Multipleofthree\} = \{3,6\}$$

$$B = \{Oddnumber\} = \{1,3,5\}$$

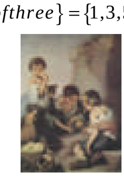
$$A \cup B = \{OddnumberORMultipleofthree\} = \{1,3,5,6\}$$

$$P(A) = 2/6$$

$$P(B) = 3/6$$

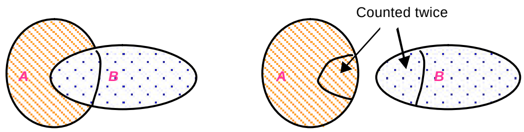
$$P(A \cup B) = 4/6$$

$$P(A) + P(B) = 5/6$$

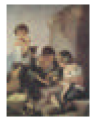


Intersection

- How to deal with the intersection
  - Note how many times the areas are counted

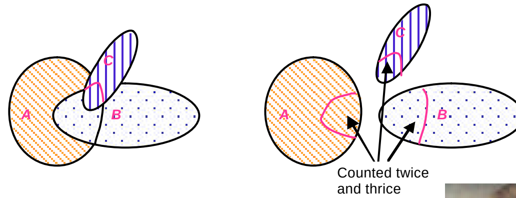


$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

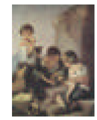


Intersection

- Case of three classes (Inclusion-Exclusion)

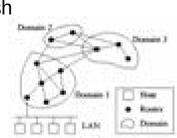


$$P(A \cup B \cup C) = P(A) + P(B) + P(B)$$
$$- P(A \cap B) - P(A \cap C) - P(C \cap B)$$
$$+ P(A \cap B \cap C)$$



Exercices

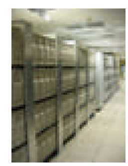
- Given traffic in internet, (i.e. packets)
  - Model the flow in a link of the mesh
  - Model the topology
  - Model queues at a node
- Define for each model
  - Sample space
  - Relevant Events
    - Note: propositions->sets->probabilities
  - Define a probability function



(Faloutsos FaloutsosandFaloutsos1999)

Exercices

- Given a computer cluster
  - Model the flow in a multi -thread program
  - Model the topology of the cluster
  - Model access to memories
  - Distributed file system
- Define for each model
  - Sample space
  - Relevant Events
    - Note: propositions->sets->probabilities
  - Define a probability function



http://en.wikipedia.org/wiki/Computer_cluster

Exercices

- Dice game:
  - Throw two dice. Wins if at least a 6 appears.
    - Sample space
    - Enumerate relevant Events
    - Compute the probability of winning

Georges de La Tour The Dice Players"

Exercices

- Lotto 6/49:
  - Sample space
  - Compute probability of relevant Events
    - i.e. Prob={3,4,5,6}
  - Compute earnings if you bought all the tickets.
  - Compute the probability of losing an euro

Georges de La Tour The Dice Players"