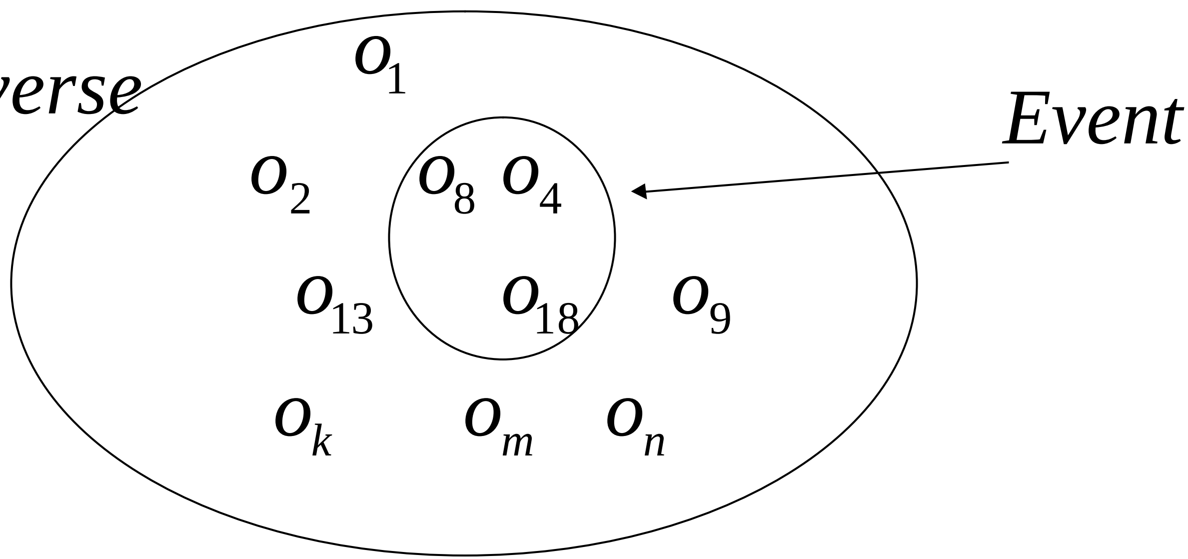
Mathematical Model of Probability

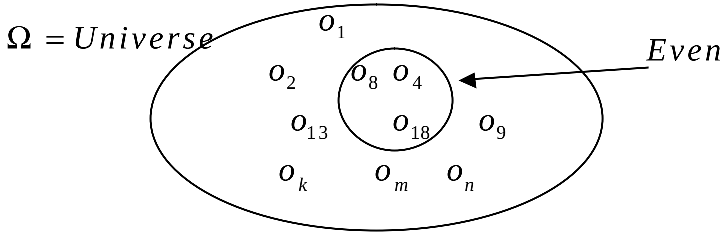
First concepts

- Sample space $\Omega = \{o_1, o_2, \cdots, o_n\}$
- Outcomes $\{o_i\}$
- Events $A = \{o_{i_1}, o_{i_2}, \cdots, o_{i_k}\}$

$$\Omega = Universe$$



First concepts



- Examples:
  - Toss a coin $\Omega = \{H, F\}$
  - Toss a coin until a F

$$\Omega = \{H, HF, FH, FF, HH, HHF, HFH \cdots, H_{\ldots}HF, \cdots\}$$

$$A = \{H, HF, HHF, \cdots, H_{\ldots}HF, \cdots\}$$

  - D'Alemberts experiment $\Omega = \{HF, FH, FF, HH\}$
    - at least one head in two tosses $A = \{HF, FH, FF\}$

Logic of Events

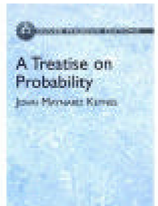
- Levels of concepts
  - Hierarchy of opperations

$$Logic = \{OR, AND, NOT, IMPLICATION\}$$

$$Sets = \{UNION, INTERSECTION, COMPLEMENT, INCLUSION\}$$

$$Sets = \{SUM, MULTIPLICATION, CONDITIONING\ (p(./.))\}$$

  - Travel through the hierarchy from prepositions to probabilities



A Treatise on Probability

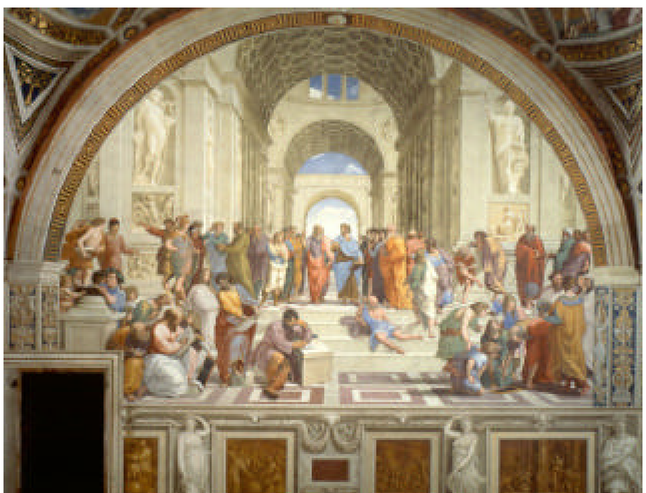
by *John Maynard Keynes*

Logic of Events

- Scope of each concept
  - Hierarchy of opperations
  - Logic Sets Probabilities

Propositions → Relations between objects → Numbers

Valid argument → Plausible Argument



Images taken from the wikipedia

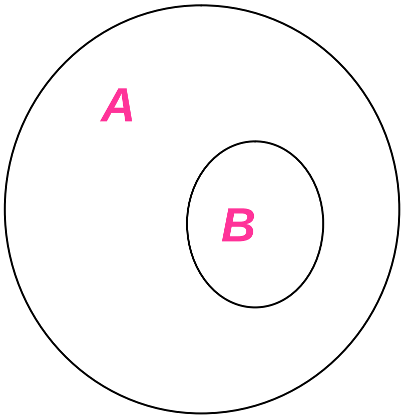
Logic of Events

Operations

Implication	$\rightarrow$	Inclusion	$\rightarrow$	Condition

- Preposition
  - If it rains, I'll bring the umbrella
- Sets
  - A={It rains}, B={Bring umbrella}

    $B \subset A$

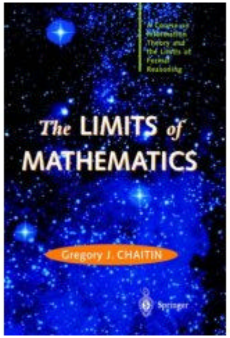


- Probabilities
  - P(B/A)=P(B)
  - P(B/Not A)= $\varnothing$

Propositions $\rightarrow$ Relations between objects $\rightarrow$ Numbers

Comments

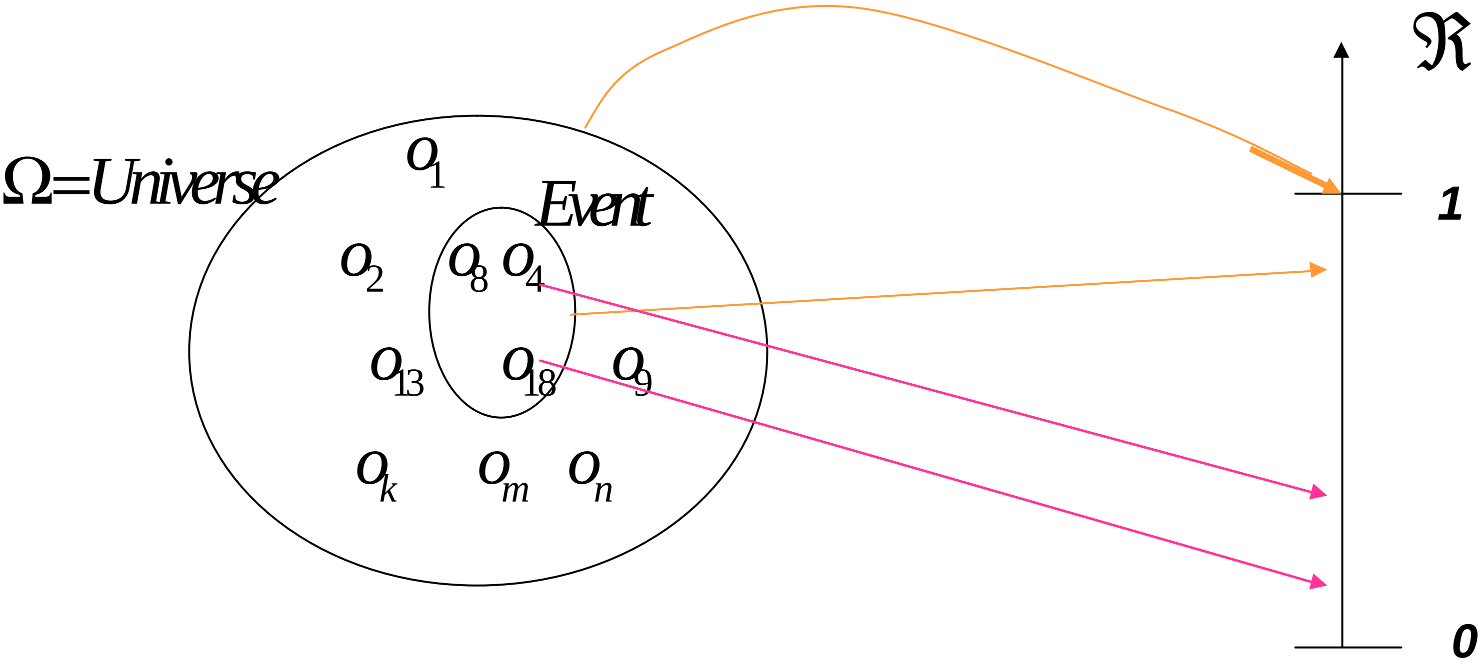
- Logic: art of thinking clearly
  - Transfer the clearness into plausible arguments.
- About Logical paradoxes and probability



The Limits of Mathematics (Discrete Mathematics and Theoretical Computer Science)
by Gregory, J. Chaitin (Hardcover - Oct 28, 2002)

The concept of probability

- Mapping $P(.):\Omega \rightarrow [0,1]$



Note: Try guessing properties of *p(.)*

The concept of probability

- Properties that should have a probability function
  - Disjoint or incompatible events

$$\text{If } B \cap A = 0 \text{ then } \mathrm{P}\left(B \cap A\right) = \mathrm{P}\left(A\right) + \mathrm{P}\left(B\right)$$

  - Sure event and impossible event

$$\mathrm{P}\left(\Omega\right) = 1 \quad \mathrm{P}\left(\varnothing\right) = 0$$

- Def. ***Probability Space*** $\left(\Omega, \mathrm{P}\right)$
  - Universe of objects plus operations
  - Note similarities with an algebra.

The concept of probability

- How to construct: $P(.):\Omega \rightarrow [0,1]$
  - Coherence
  - From structure or empirical measurement?
- D'Alembert's problem (Ω, P)

$$\Omega = \{HF, FH, FF, HH\}$$

$$P(\{HF\}) = P(\{FH\}) = P(\{FF\}) = P(\{HH\}) = 1/4$$

$$P(\{HF, FH\}) = \cdots = P(\{FF, HH\}) = 1/2$$

$$P(\{HF, FH, FF\}) = \cdots = P(\{FH, FF, HH\}) = 3/4$$

$$P(\{HF, FH, FF, HH\}) = 1$$

The concept of probability

- How to construct: $P(.):\Omega \rightarrow [0,1]$
  - Compute areas
  - Prob. a special case of *Measure theory*.

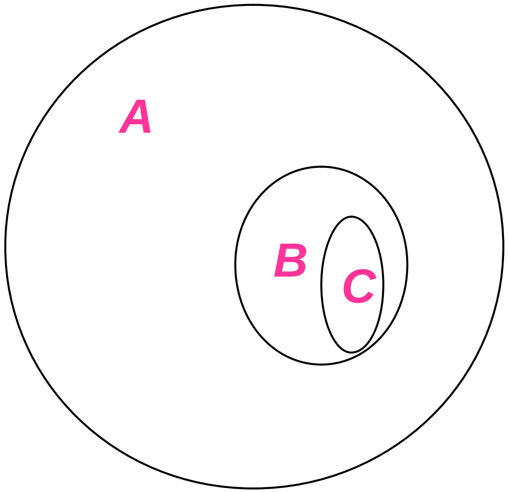
$$\mathrm{P}(\Omega) = Total\ Area = 1$$

$$\mathrm{P}(A) = Area\ of\ A$$

$$\mathrm{P}(C) = Area\ of\ C$$

$$\mathrm{P}(C/B) = \frac{Area\ of\ C}{Area\ of\ B}$$

$$\mathrm{P}(B/C) = Area\ of\ C$$



Intersection

- How to deal with the intersection
  - Example: Die $\Omega = \{1, 2, 3, 4, 5, 6\}$
    - Set of events

$$A = \{Multiple\ of\ three\} = \{3, 6\}$$

$$B = \{Odd\ number\} = \{1, 3, 5\}$$

$$A \cup B = \{Odd\ number\ OR\ Multiple\ of\ three\} = \{1, 3, 5, 6\}$$

$$P(A) = 2/6$$

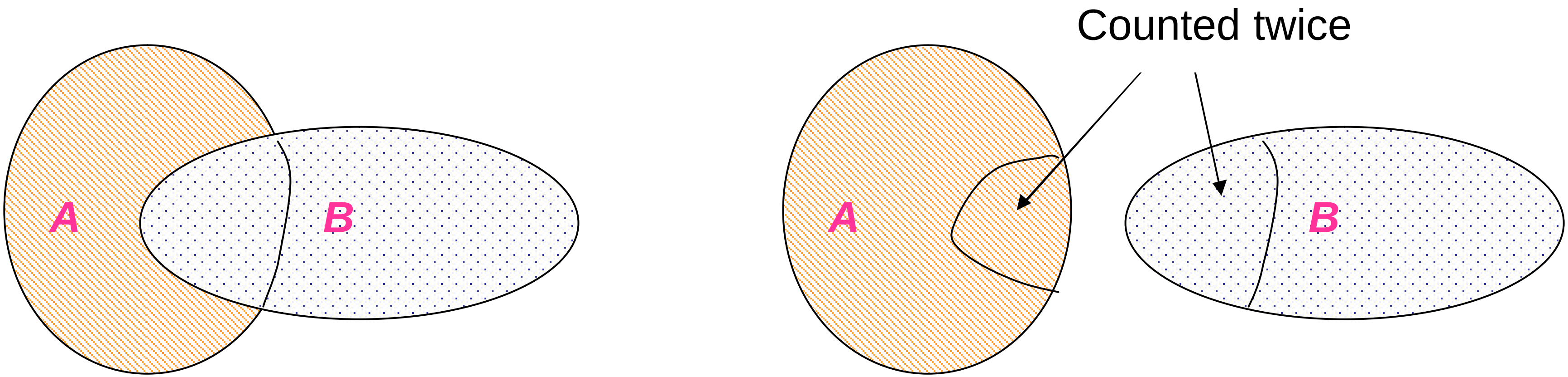
$$P(B) = 3/6$$

$$P(A \cup B) = 4/6$$

$$P(A) + P(B) = 5/6$$

Intersection

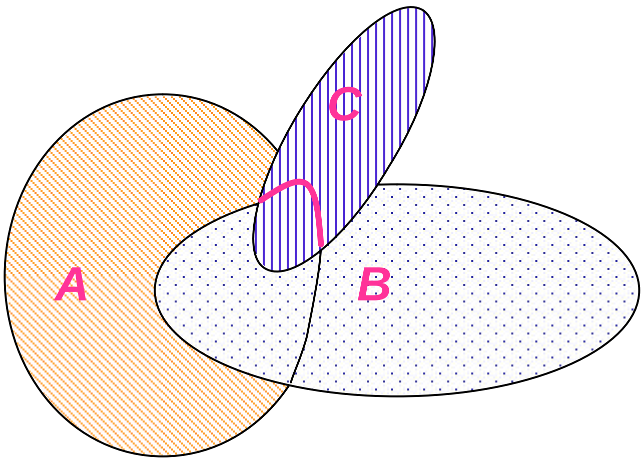
- How to deal with the intersection
  - Note how many times the areas are counted

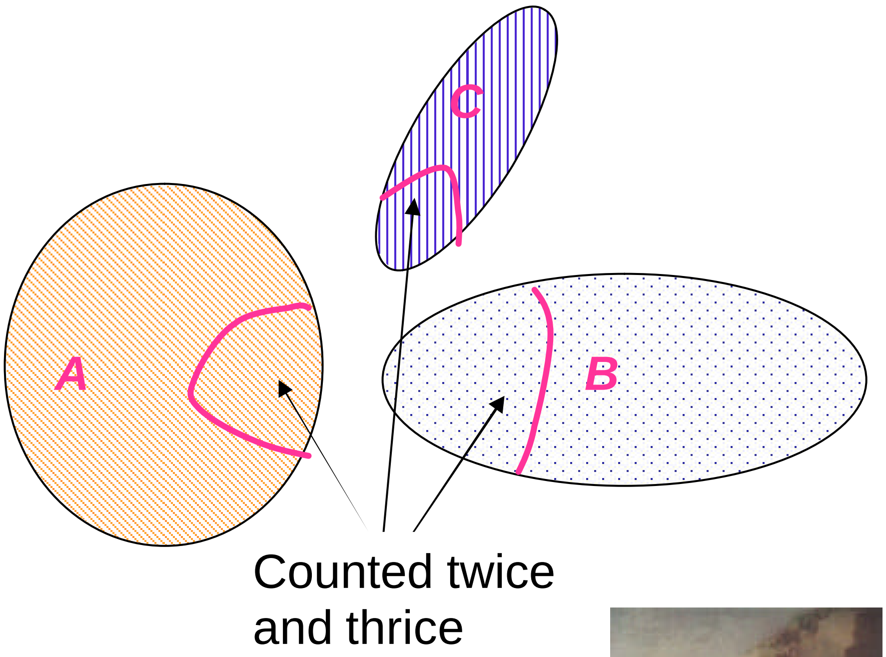


$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Intersection

- Case of three classes (Inclusion-Exclusion)

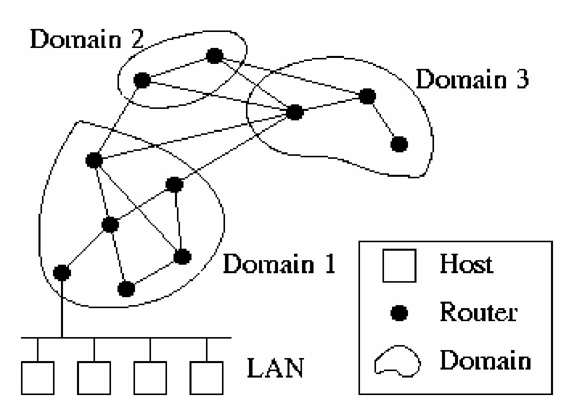




$$P(A \cup B \cup C) = P(A) + P(B) + P(B)$$
$$- P(A \cap B) - P(A \cap C) - P(C \cap B)$$
$$+ P(A \cap B \cap C)$$

Exercices

- Given traffic in internet, (i.e. packets)
  - Model the flow in a link of the mesh
  - Model the topology
  - Model queues at a node
- Define for each model
  - Sample space
  - Relevant Events
    - Note: propositions->sets->probabilities
  - Define a probability function



(Faloutsos, Faloutsos and Faloutsos, 1999)

Exercices

- Given a computer cluster
  - Model the flow in a multi-thread program
  - Model the topology of the cluster
  - Model access to memories
  - Distributed file system
- Define for each model
  - Sample space
  - Relevant Events
    - Note: propositions->sets->probabilities
  - Define a probability function

http://en.wikipedia.org/wiki/Computer_cluster

Exercices

- Dice game:
  - Throw two dice. Wins if at least a 6 appears.
    - Sample space
    - Enumerate relevant Events
    - Compute the probability of winning

Georges de La Tour The Dice Players"

Exercices

- Lotto 6/49:
  - Sample space
  - Compute probability of relevant Events
    - i.e. Prob={3,4,5,6}
  - Compute earnings if you bought all the tickets.
  - Compute the probability of losing an euro