Independent Events
Two definitions of independence

• Def. 1
  – Two events, A and B are said to be independent if
    \[ P(A \cap B) = P(A)P(B) \]

• Def. 2
  – Two events, A and B are said to be independent if
    \[ P(A/B) = P(A) \]

• Note that they are algebraically equivalent

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)}
\]
Intuitive meaning of independence

- $P(A / B) = P(A)$
  - Knowledge of $B$ is irrelevant to $A$
    - $P(\text{Thunder/lightning}) \neq P(\text{Thunder})$
    - $P(\text{Face coin1/Face coin2}) = P(\text{Face coin1})$
  - Sample space of $A$ does not change if $B$ has happened.
    - For instance a sample space generated by the cartesian product of two sets.
      \[
      \Omega_1 = \{ A_1, A_2, \ldots A_n \}
      \]
      \[
      \Omega_2 = \{ B_1, B_2, \ldots B_m \}
      \]
      \[
      \Omega = \Omega_1 \times \Omega_2
      \]
      \[
      \Omega = \{ A_1 B_1, A_1 B_2, \ldots A_1 B_m, \ldots A_n B_m \}
      \]
Intuitive meaning of independence

- Sample space of A does not change if B has happened.
  - Sample space generated by the cartesian product of two sets.

<table>
<thead>
<tr>
<th>$\Omega_1$</th>
<th>$\Omega_2$</th>
<th>$\Omega = \Omega_1 \times \Omega_2$</th>
</tr>
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<tbody>
<tr>
<td>$A_1$</td>
<td>$B_1$</td>
<td>$A_1 B_1$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$B_2$</td>
<td>$A_1 B_2$</td>
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<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$A_n$</td>
<td>$B_m$</td>
<td>$A_n B_m$</td>
</tr>
</tbody>
</table>

- $P(A/B) = P(A)$
- $P(A \cap B) = P(A)P(B)$
- $p(A_1) = \frac{1}{n}$
- $p(A_1 B_1) = \frac{1}{nm}$
- $p(B_1) = \frac{1}{m}$
- $p(A_1 / B_1) = \frac{1}{n}$
Explaination of dependent events by means of the sample space

- Sample space of A does change if B has happened. Eliminate possibilities

\[
P(A|B) \neq P(A) \\
P(A \cap B) \neq P(A)P(B)
\]

\[
\begin{align*}
p(A_1) &= \frac{1}{n} \\
p(A_1B_1) &= \frac{1}{nm-1} \\
p(B_1) &= \frac{1}{m} \\
p(A_1 / B_1) &= \frac{m}{nm-1}
\end{align*}
\]
Explaination of dependent events by means of the sample space

• Sample space of A **does change** if B has happened.
  – Eliminated possibilities
  – Preferencial Attatchment

Model 1 of the problem
A1=Rain, A2=Sun shine
B1=Thunder

Model 2 of the problem
A1=Rain, A2=Sun shine
B1=Dressed with a rain coat

<table>
<thead>
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<th>$\Omega_1$</th>
<th>$\Omega_2$</th>
<th>$\Omega_{\text{New}}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$A_1B_1$</td>
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<td>$B_2$</td>
<td>$A_1B_2$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
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</tr>
<tr>
<td>$A_n$</td>
<td>$B_m$</td>
<td>$A_nB_m$</td>
</tr>
</tbody>
</table>

$P(A/B) \neq P(A)$
$P(A \cap B) \neq P(A)P(B)$

$P(A_i) = \frac{1}{n}$
$P(A_iB_1) = \frac{1}{nm - 1}$
$P(B_1) = \frac{1}{m}$
$P(A_i / B_1) = \frac{m}{nm - 1}$
Intuitive meaning of independence

• Another case: **Proportion** of the sample space of A does not change if B has happened
  
  • Note: the condition is algebraic, not **physical**

\[
P(S_1) = 1/2 \\
P(S_2) = 5/12
\]

\[
P(\text{Slw}/ S_1) = \frac{1}{3} = \frac{P(\text{Slw} \cap S_1)}{P(S_1)}
\]

\[
P(\text{Slw}/ S_2) = \frac{2}{5} = \frac{P(\text{Slw} \cap S_2)}{P(S_2)}
\]

\[
P(\text{Slw}) = P(S_2)P(\text{Slw}/ S_2) + P(S_1)P(\text{Slw}/ S_1)
\]

\[
P(\text{Slw}) = \frac{1}{2} \times \frac{5}{3} + \frac{1}{2} \times \frac{2}{12} = \frac{1}{3}
\]
Independence by inclusion

Operations

<table>
<thead>
<tr>
<th>Implication</th>
<th>Inclusion</th>
<th>Condition</th>
</tr>
</thead>
</table>

• Preposition
  – If it rains, I’ll bring the umbrella

• Sets
  – $A = \{\text{It rains}\}$, $B = \{\text{Bring umbrella}\}$
    $B \subset A$

• Probabilities
  – $P(B/A) = P(B)$
  – $P(B/\text{Not } A) = \emptyset$

Propositions $\rightarrow$ Relations between objects $\rightarrow$ Numbers
Intuitive meaning of independence

- **Proportion** of the sample space of $A$ does not change if $B$ has happened
  - Note: the condition is algebraic, not physical

\[
P(\Omega) = \text{Total Area} = 1
\]

\[
P(A) = \frac{\text{Yellow Area}}{\text{Total Area}}
\]

\[
P(B) = \frac{\text{Blue Area}}{\text{Total Area}}
\]

\[
P(A/B) = \frac{\text{Green Area}}{\text{Blue Area}} = \frac{\text{Yellow Area}}{\text{Total Area}}
\]

\[
P(A/B) = P(A)
\]
Application to Scale Free objects

• Application to fractal images and objects.
  – Sierpinski triangle

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Application to Scale Free objects

- Application to fractal images and objects.

http://en.wikipedia.org/wiki/Fractals

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P(A/B) = \frac{\text{Green Area}}{\text{Blue Area}} = \frac{\text{Yellow Area}}{\text{Total Area}}
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\[
P(A/B) = P(A)
\]
Application to Scale Free objects

- Application to internet traffic.

\[ A = \{ q \% \text{ change in the traffic} \} \]
\[ B_0 = \{ \text{time scale: month} \} \]
\[ B_1 = \{ \text{time scale: day} \} \]
\[ B_2 = \{ \text{time scale: hour} \} \]
\[ B_3 = \{ \text{time scale: seconds} \} \]
\[ \forall \ i, j \]
\[ P(A) = P(A/B_i) \]
\[ P(A/B_j) = P(A/B_i) \]

http://classes.yale.edu/fractals/Panorama/ManuFractals/Internet/Internet4.html
Application to Scale Free objects

- Flips of coins. 10,000 vs. 1,000,000

\[ \Omega_1 = \{ \text{set of all possible results in 10,000 flips of a coin} \} \]
\[ \Omega_2 = \{ \text{set of all possible results in 1,000,000 flips of a coin} \} \]
\[ A = \{ \text{Fraction of Time one player is winning} \} \]
\[ B = \{ \text{Scale of the experiment} \} \]

\[ p(A) = p(A / B) \]
Application to Scale Free objects

• One way of creating Scale free objects, is by means of an exponential growth.
Application to Scale Free objects

Prefencial connexion (road to the nearest neighbour) vs. indifferent connexion (can fly anywhere)

\[
\Omega_1 = \{ A_1, A_2, \ldots A_n \}
\]

\[
\Omega_2 = \{ B_1, B_2, \ldots B_m \}
\]

\[
\Omega = \Omega_1 \times \Omega_2
\]

\[
\Omega = \{ A_1 B_1, A_1 B_2, \ldots A_1 B_m, \ldots A_n B_m \}
\]

Taken from:

*The architecture of complexity:
From the topology of the www to the cell’s genetic network*

Albert-László Barabási
Similarities between natural graphs

• Semantic map vs. Physical connections in internet
Examples of Scale Free in biology

- Broccoli
- Eucalyptus Tree
Relation between independence and disjoint condition

• Independence does not imply disjointness
  – Condition of indepence \[ P(A \cap B) = P(A)P(B) \]
  – Condition of disjointness \[ A \cap B = 0 \]
  • In probabilities means:
    \[
    P(A \cup B) = P(A) + P(B) - P(A \cap B)
    \]
    \[
    P(A \cup B) = P(A) + P(B)
    \]

• What does \[ P(A \cap B) = P(A)P(B) = 0 \] mean?
Relation between independence and disjoint condition

- What does \( P(A \cap B) = P(A)P(B) = 0 \) mean?

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

Eliminated possibilities
Preferencial Attatchment

Model 1 of the problem
A1=Rain, A2=Sun shine
B1=Thunder

Model 2 of the problem
A1=Rain, A2=Sun shine
B1=Dressed with a rain coat
Probability of the intersection of a set of independent events.

- Probability of the union of independent events
  \[ \Omega = \{ A_1, A_2, \ldots, A_n \} \]
- Formally the union of all the elements, consists on the event:
  - \( E = \{ \text{Simultaneously of the elements of the set appear} \} \)

\[ P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i) \]

- Note:
  Propositions → Relations between objects → Numbers
When intersection of sets corresponds to multiplication of probabilities?

\[ P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i) \]

Propositions → Relations between objects → Numbers

\[ \text{Logic} = \{ \text{OR, AND, NOT, IMPLICATION} \} \]
\[ \text{Sets} = \{ \text{UNION, INTERSECTION, COMPLEMENT, INCLUSION} \} \]
\[ \text{Sets} = \{ \text{SUM, MULTIPLICATION, CONDITIONING (} p(./.) \text{)} \} \]
Probability of getting at least one event of a set of independent events

• Probability of the union of independent events \( \Omega = \{ A_1, A_2, \cdots A_n \} \)

• Formally the union of all the elements, consists on the event:
  – \( E = \{ \text{At least one of the elements of the set appear} \} \)
  – \( \bar{E} = \{ \text{Not a single element of the set appears} \} \)

• Which is equivalent to \( E = \{ \Omega - \bar{E} \} \)
Probability of getting at least one event of a set of independent events

- Probability of the union of independent events

\[ E = \bigcup_{i=1}^{n} A_i \]

\[ E = \{ \text{At least one of the elements of the set appear.} \} \]

\[ \bar{E} = \bigcap_{i=1}^{n} (\Omega - A_i) \]

\[ \bar{E} = \{ \text{Not a single element of the set appears} \} \]

\[ E = \Omega - \bigcap_{i=1}^{n} (\Omega - A_i) \]

\[ P(E) = P\left(\Omega - \bigcap_{i=1}^{n} (\Omega - A_i)\right) = 1 - P\left(\bigcap_{i=1}^{n} (\Omega - A_i)\right) = \]

\[ P(E) = 1 - \prod_{i=1}^{n} [1 - P(A_i)] \]
Example 1

- A web page has two kind links. \{A, B\}
- \(M\) different users select randomly and independently of each other one of the links.
- What is the probability that at a link of kind A is visited least once?

- For instance: Web based bookshop that also has CD, DVD, second hand books.
Example 1

- A web page has two kind links. \{A,B\}
- Sample space of the links

\[ \Omega_1 = \{A_1, A_2, \ldots A_n\} \quad P(A) = \frac{n}{n+m} \]
\[ \Omega_2 = \{B_1, B_2, \ldots B_m\} \quad P(B) = \frac{m}{n+m} \]

- Possible choises of the \(M\) users

Possible of choices = \((\{A_{i_1} \ OR \ B_{j_1}\}, \{A_{i_2} \ OR \ B_{j_2}\}, \ldots \{A_{i_M} \ OR \ B_{j_M}\})\)

Number of choices = \(2 \times 2 \times \ldots 2 = 2^M\)
Example 1

• Probability of a given selection:

\[
P(\{A_{i_1} \text{ AND } A_{i_2} \text{ AND } B_{j_1} \ldots A_{i_L} \text{ AND } B_{j_{M-L}} \}) = \left( \frac{n}{n+m} \right)^L \left( \frac{m}{n+m} \right)^{M-L}
\]

\[
P(A) = \frac{n}{n+m}
\]

\[
P(B) = \frac{m}{n+m}
\]

• What is the probability that at a link of kind A is visited least once?

\[
P(\{\text{At least an } A\}) = 1 - P(\{\text{All B}\}) = 1 - \left( \frac{m}{n+m} \right)^M
\]
Example 1

• What is the probability that at a link of kind A is visited least once?

\[
P(\text{At least an } A) = 1 - P(\text{All } B) = 1 - \left( \frac{m}{n + m} \right)^M
\]

\[
P(A) = \frac{n}{n + m}
\]

\[
P(B) = \frac{m}{n + m}
\]

<table>
<thead>
<tr>
<th>( M )</th>
<th>( m=2 )</th>
<th>( n=3 )</th>
<th>( P(A)= )</th>
<th>( P(B)= )</th>
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<td>0,4</td>
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<table>
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<th>( P(B)= )</th>
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</tbody>
</table>

Graph for \( m=2 \) and \( n=3 \):

Graph for \( m=10 \) and \( n=3 \):
Example 2

• Another way of deriving the formula:

\[ P(\{ \text{At least an A} \}) = 1 - P(\{ \text{All B} \}) = 1 - \left( \frac{m}{n+m} \right)^M \]

• Throw a coin N times, what is the probability that heads occur on at least one trial?

\[ P(\{ \text{Heads at least in one trial} \}) = p + q^2 p + q^3 p \cdots + q^{M-1} p = p \frac{1-q^M}{1-q} = 1 - q^M \]

How?
Example 2

• Throw a coin N times, what is the probability that heads occur on at least one trial?

\[ A_i = \{\text{First Head occurs in the trial number } i\} \]
\[ A_i = \{(i-1) \text{ Tails followed by a Head} \cup \text{ (then anything else)}\} \]

\[ P(A_i) = q^{i-1} p + P(\{\text{then anything else}\}) = q^{i-1} p \]

\[ P(\{\text{Heads at least in one trial}\}) = P(A_1 \cup A_2 \cdots \cup A_M) = \sum_{i=1}^{M} A_i \]

\[ P(\{\text{Heads at least in one trial}\}) = p + q^2 p + q^3 p \cdots + q^{M-1} p = p \frac{1 - q^M}{1 - q} = 1 - q^M \]
Example 2

- $P(\{\text{then anything else}\})$?

Case of 3

$P(\{\text{then anything else}\}) = ppp + \underbrace{ppq + pqp + qpp + qqp + qpq + pqq + qqq}$

$$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + b^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

$\{aab, aba, baa\}$

$$(p + q)^n = 1$$
Example 2

• Applications of the formula:

\[ P(\{\text{At least an A}\}) = 1 - P(\{\text{All B}\}) = 1 - \left( \frac{m}{n+m} \right)^M \]

• Carl Sagan on the probability of intelligent life in our galaxy

• Saddam’s ‘Plebicito’ with a 99.9% of approval

• Other ‘plebicito’s’ and elections