Allocation of probabilities

Allocation of probabilites

- Construct a bridge between reality and the mathematical model of probability

Real world → Model describing the Real World →  → $P(A)=2/6$, $P(B)=3/6$, $P(A\cup B)=4/6$

$$\Omega=\{1,2,3,4,5,6\}$$

$$A=\{Multiple\ of\ \ three\}=\{3,6\}$$

$$B=\{Odd\ \ number\}=\{1,3,5\}$$

$$A\cup B=\{Odd\ \ number\ OR\ Multiple\ of\ \ three\}=\{1,3,5,6\}$$

FAIR DIE (?)

Laplace's Rule

- Premise on Laplace's Rule:
- Principle of insufficient reason
  - If we do not know the probabilities, we should suppose that all possibilities are equaly likely.
  - Problem: the probabilities depend on how a sample space is **defined**.

$$\Omega = \{She-Cat, He-Cat\} \to \{1/2, 1/2\}$$

$$\Omega^{'} = \{She-Cat, He-Cat, I-don't-know\} \to \{1/3, 1/3, 1/3\}$$

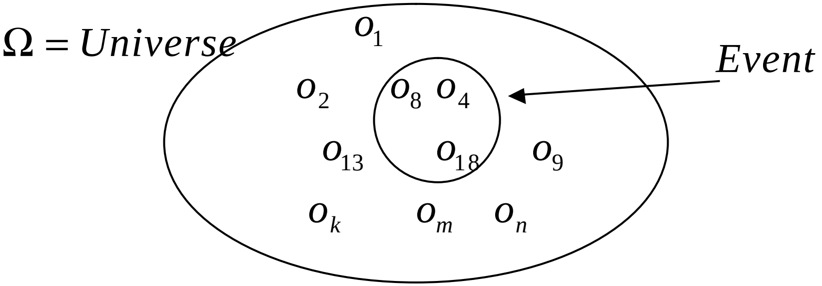
  - Important in *Decision Theory*. Savage Principles of Statistics

Laplace's Rule

- Laplace's Rule:

$$\Pr = \frac{Count\ of\ ways\ for\ a\ result}{Count\ of\ all\ possible\ results}$$

  – If **all** possible cases have the **same** probability.



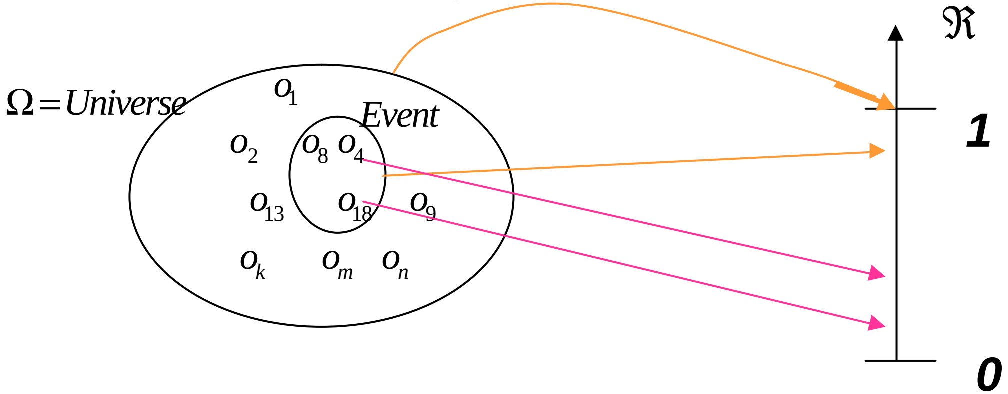
Laplace's Rule

- Problem of *self-reference*:

$$\Pr = \frac{Count\ of\ ways\ for\ a\ result}{Count\ of\ all\ possible\ results}$$

  – If **all** possible cases have the **same** probability.

Somewhere we have to introduce an arbitrary selection



Example:

FAIR DIE (?)

Laplace's Rule

- If we know that an outcome is more possible than another $i.e.\ o_2$ we create new labels for the outcome.

$$\Omega = \{o_1, o_2, \cdots, o_n\} \rightarrow \Omega' = \{o_1, o_2^1, o_2^2, \cdots o_2^m, \cdots, o_n\}$$

$$\Pr = \frac{Count\ of\ ways\ for\ a\ result}{Count\ of\ all\ possible\ results}$$

Example 1

- A bag with 6 balls.
  - 3 Red, 2 Green, 1 white.
  - Compute the prob. of extracting each colour.

$$\Omega = \{R_1, R_2, R_3, V_1, V_2, W_1\}$$

$$Events$$

$$A = \{R_1, R_2, R_3\}$$

$$B = \{V_1, V_2\}$$

$$C = \{W_1\}$$

$$\Pr(A) = \frac{3}{6}$$

$$\Pr(B) = \frac{2}{6}$$

$$\Pr(C) = \frac{1}{6}$$

Example taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

Example 2

- An Urn contains three cards, one with both sides red, one with both sides white, and a third with one side red and the other white.
- We extract a card, and examine ***only one*** side.
- Compute the probability that the ***unseen side*** is of the ***same*** colour.
  - Mistaken line of reasoning (D'Alembert's mistake)
  - The unseen *outcome* is that "side can be of equal or different to the observed"
  - This implies that both results have the same probability, and that the sample space is $\Omega = \{equal, different\}$

Example taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

Example 2

- Real sample space. Take into account that the outcome is to set a given face upwards.
  - Name each face

$$\Omega = \left\{ R_1R_2 , R_2R_1 , R_3W_3 , W_3R_3 , W_1W_2 , W_2W_1 \right\}$$

- If each outcome is equally likely->pr=1/6
- Event={two faces with the same colour}

$$A = \left\{ R_1R_2 , R_2R_1 , W_1W_2 , W_2W_1 \right\} \qquad \Pr(A) = \frac{4}{6}$$

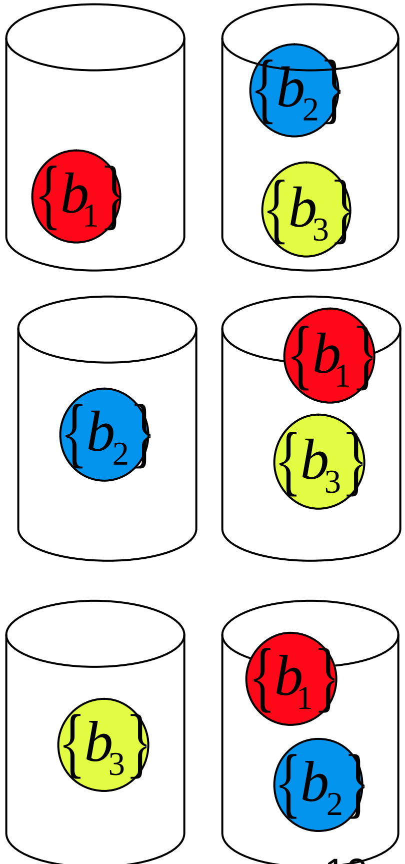
$$B = \left\{ R_3W_3 , W_3R_3 \right\} \qquad \Pr(B) = \frac{2}{6}$$

Two out of three will be equal

Example taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

Example 3

- Three balls are put in two urns {*I*,*D*}.
- Probability that *I* has at least a ball?
  - Mistake:
  - Possible sample space $\Omega = \{0,1,2,3\}$
  - Note: there is only one way of introducing 3 balls in {*I*}
  - if there is only one, it could be the first, second or third.



Example taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

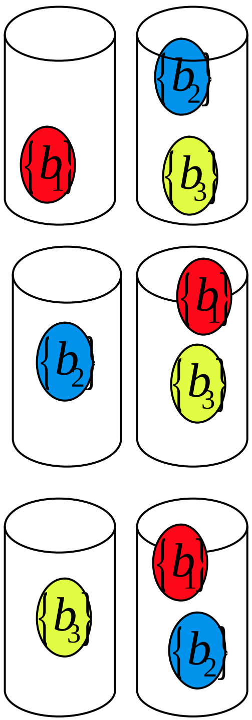
Example 3

- Distinguish the ball by the introduction moment

  $$\{b_1, b_2, b_3\}$$

  - Sample space

$$\Omega = \{III, IID, IDI, IDD, DII, DID, DDI, DDD\}$$



  - Now each outcome has the probability->1/8

  - Event A={At least one ball in *I*}

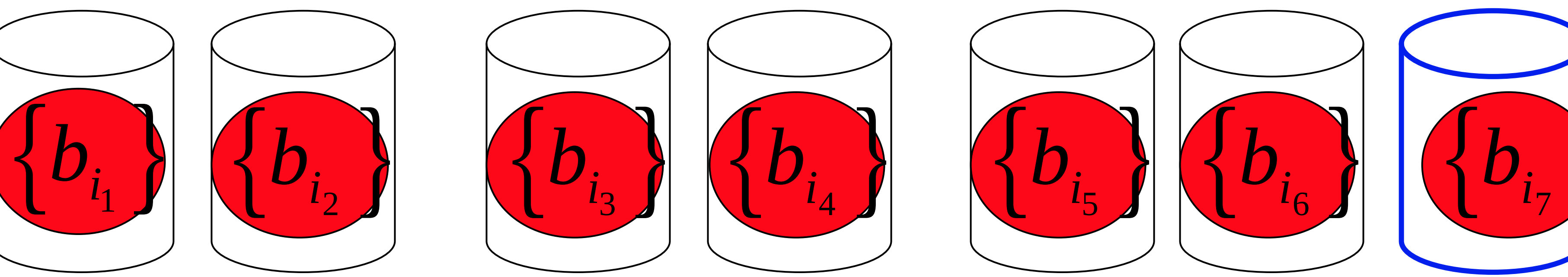
$$A = \{III, IID, IDI, IDD, DII, DID, DDI\}$$

$$\Pr(A) = \frac{7}{8}$$

Combinatorial Methods.Lotto6/49

- Lotto6/49: 6 numbers+ 1 complementary are selected from 49.
  - Probability of guessing k
  - Probability of guessing k AND the complentary
  - Probability of guessing k AND Not the complentary

$$\left\{b_1, b_2, b_3, \cdots, b_{48}, b_{49}\right\} \rightarrow \{i_1, i_2, \cdots, i_7\} \rightarrow \{b_{i_1}, b_{i_2}, \cdots, b_{i_7}\}$$



Example taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

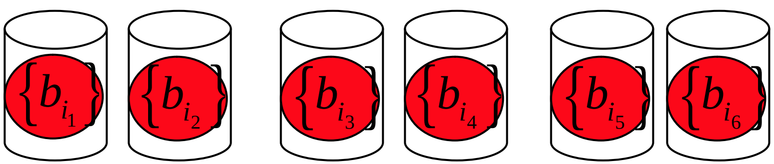
Combinatorial Methods.Lotto6/49

- Number of ways for guessing n results.

$$\binom{6}{n}\binom{43}{6-n}$$

Different sets with the winning numbers

$$\{i_1, i_2, \cdots, i_n\}$$



Different sets with the lossing numbers

$$\{b_1, b_2, b_3, \cdots, b_{48}, b_{49}\} - \{b_{i_1}, b_{i_2}, \cdots, b_{i_n}$$

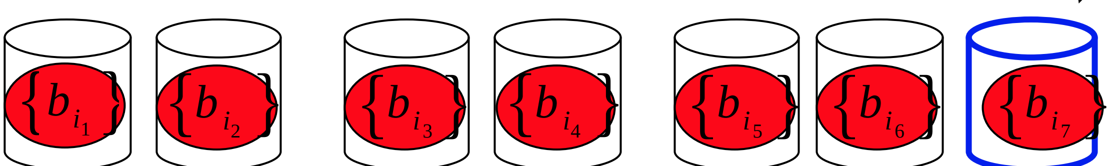
Combinatorial Methods.Lotto6/49

- Probability of guessing k

$$\Pr(n)=\frac{\binom{43}{6-n}\binom{6}{n}}{\binom{49}{6}}$$

- Probability of guessing k
  AND the complentary

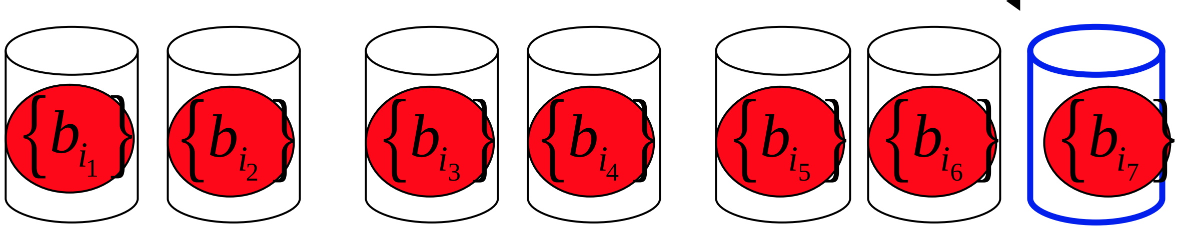
$$\Pr(n)=\frac{\binom{43}{6-n}\binom{6}{n}\binom{43-n}{1}}{\binom{49}{6}}=\frac{\binom{43}{6-n}\binom{6}{n}}{\binom{49}{6}}(43-n)$$



Combinatorial Methods.Lotto6/49

- Probability of guessing k
  AND NOT the complentary

$$\Pr(n) = \frac{\binom{43}{6-n}\binom{6}{n}\binom{43-(6-n)}{1}}{\binom{49}{6}} = \frac{\binom{43}{6-n}\binom{6}{n}}{\binom{49}{6}}(37+n)$$



Combinatorial Methods. Coincidences

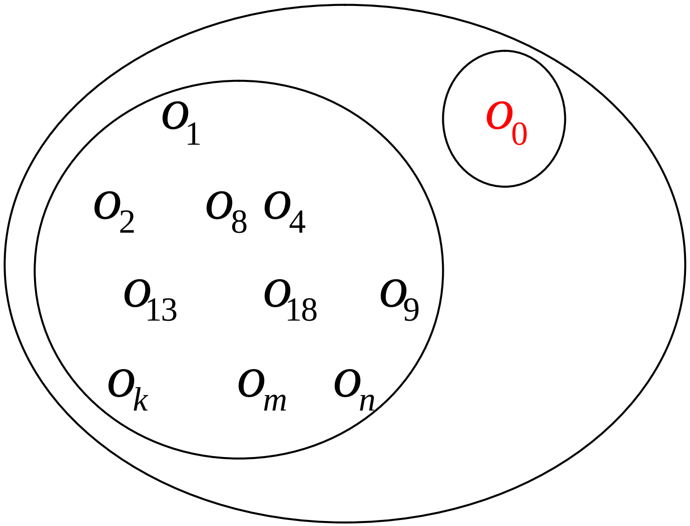
- In a planet with years of N days, we have *n* habitants $n<N$.
  - Probability that at least two have the birthday at the same day.
  - Value of *n* in so that the probability of coincidence is ½
    - Note: At-least-one=one OR two OR…..

Example taken from VÉLEZ , HERNÁNDEZ, Cálculo de Probabilidades

Combinatorial Methods. Coincidences

- Note:
  - ***A***={ **At-least-one**}={one OR two OR…..}

$$\Pr\left(\Omega - \{o_0\}\right) = \Pr\left(o_1\right) + \Pr\left(o_2\right) + \Pr\left(o_3\right) + \cdots = 1 - \Pr\left(o_0\right)$$

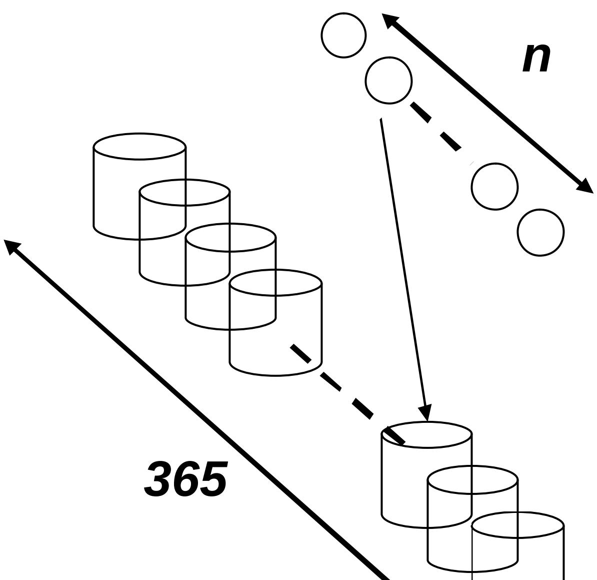


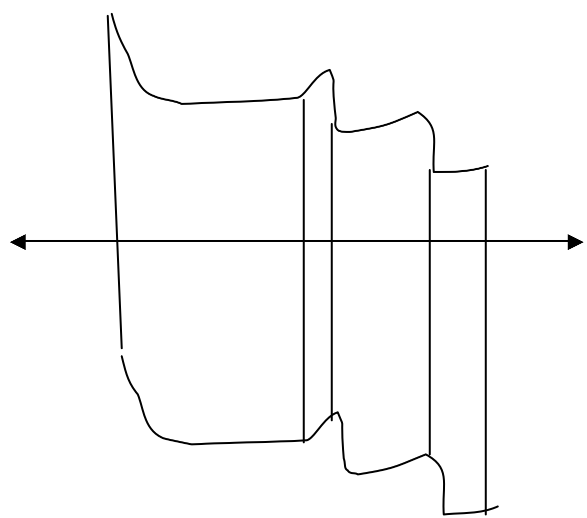
Strategy: We will compute the probability that there is no coincidence

Combinatorial Methods. Coincidences

- Possible ways of having no coincidence

$$\underbrace{365}_{N}\underbrace{364}_{N-1}\underbrace{363}_{N-2}\cdots\underbrace{365-(n-2)}_{N-(n-2)}\underbrace{365-(n-1)}_{N-(n-1)}$$





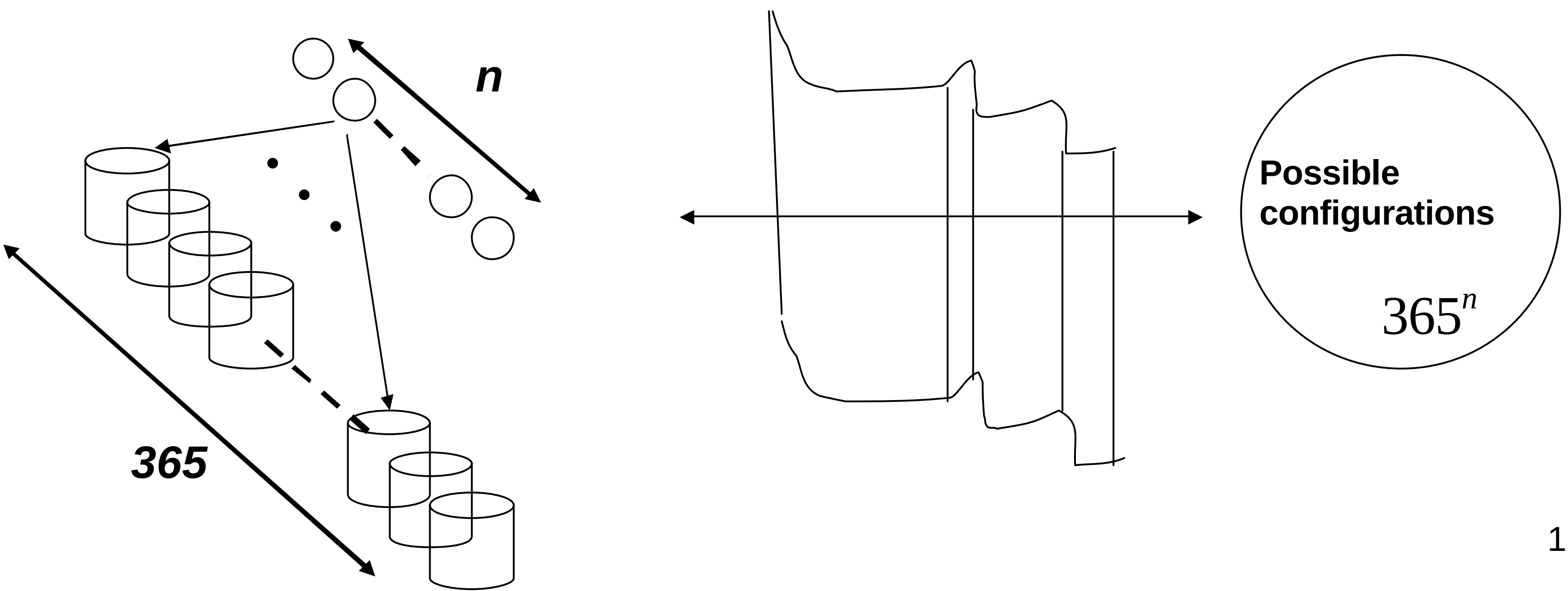
Possible configurations

$$\frac{N!}{(N-n)!}$$

Combinatorial Methods. Coincidences

- Possible ways of allocation of birthdays

$$\underbrace{NNN\cdots N}_{n} = N^n$$



Combinatorial Methods. Coincidences

- Final probability:

$$\Pr(A) = 1 - \frac{N!}{(N-n)!}\frac{1}{N^n}$$

n	5	10	15	20	25	50
p	0.027	0.117	0.253	0.411	0.569	0.97

- From Paulo’s Book:
  - TV show, the presenter bets that someone in the public (>50) has the same birthday.
  - What is the mistake?

Combinatorial Methods. Coincidences

- Final probability:

$$\Pr(A) = 1 - \frac{N!}{(N-n)!}\frac{1}{N^n}$$

  - Can be difficult to compute.
  - We substitute by the Stirling's aprox.

$$N! = \sqrt{2p}\, N^{N+1/2} e^{-N}$$

$$1 - \Pr(A) = \frac{\sqrt{2p}\, N^{N+1/2} e^{-N}}{\sqrt{2p}\,(N-n)^{N-n+1/2}\, e^{n-N}}\frac{1}{N^n} = \left(1 - \frac{n}{N}\right)^{n-1/2-N} e^{-n}$$

$$\lim_{a\to\infty}\left(1 + \frac{x}{a}\right)^a = e^x$$

Combinatorial Methods. Coincidences

- Final probability: $\Pr(A) = 1 - \frac{N!}{(N-n)!}\frac{1}{N^n}$

$$1 - \Pr(A) = \left(1 - \frac{n}{N}\right)^{n-1/2-N} e^{-n}$$

$$(n - 1/2 - N)\log\left(1 - \frac{n}{N}\right) \cong (n - 1/2 - N)\left(-\frac{n}{N} - \frac{n^2}{2N^2}\right)$$

$$\log(1+p) = -p - \frac{p^2}{2} - \frac{p^3}{3} - \cdots$$

Combinatorial Methods. Coincidences

- Final probability: $\Pr(A) = 1 - \frac{N!}{(N-n)!}\frac{1}{N^n}$

$$(n-1/2-N)\log\left(1-\frac{n}{N}\right) \cong (n-1/2-N)\left(-\frac{n}{N}-\frac{n^2}{2N^2}\right) \cong n-\frac{n(n-1)}{2N}$$

$$1-\Pr(A) = \left(1-\frac{n}{N}\right)^{n-1/2-N} e^{-n} \cong e^{-\frac{n(n-1)}{2N}}$$

When we study Poisson we will get
the same expresion ☺

Combinatorial Methods

- The movie “The Quick and the Dirty” a russian rulette duel, with 6 identical shots of whiskey, one is laced with Strychnine.
- Two duelist must drink at turns.
- Bad guy offers $1000 to the good if he drinks first.
- ***Is it worth while?***

Combinatorial Methods

- Possible arrangements of the glasses=6!
- Possible arrangements if the strychnine glass is the first one=5!
- P(First Glass has Stry.)=5!/6!=1/6
  – Notice that by symetry, all have the same prob.
- Prob(A duelist drinks Stry)

$$\Pr(A) = \frac{3*5!}{6!} = \frac{1}{2}$$

Combinatorial Methods

- Prob. that any duelist drinks Strychnine

$$\Pr(A) = \frac{3 * 5!}{6!} = \frac{1}{2}$$

- If the first survives, the prob. that the bad one drinks it is:

$$\Pr(A) = \frac{3 * 4!}{5!} = \frac{3}{5}$$

Combinatorial Methods "socks problem" (*Murphy's law*)

- The "socks problem" (*Murphy's law*)
  - Take 10 different pairs of socks to the laundormat.
  - What are the probabilities of seven and four matching socks remaining when 6 socks are lost during the washing.?
  - Best scenario 7 matching pairs. Worst case 4 matching pairs.
    - Does it look much different?

Combinatorial Methods "socks problem" (*Murphy's law*)

- Possible ways to choose 6 socks of 10 pairs. $\binom{20}{6}$

- We have 7 complete pairs only if both socks of the three pairs are missing which can happen in $\binom{10}{3}$ ways

- Final probability: $\Pr = \frac{\binom{10}{3}}{\binom{20}{6}} = 0.0031$

Combinatorial Methods "socks problem" (*Murphy's law*)

- You are left with **only** 4 matching pairs.
- If only exactly one sock of each of the 6 pairs is missing, the ways for chosing 6 socks so that 4 matching socks are left is

$$\binom{10}{6}2^6 \leftarrow \left\{s_1^i, s_2^i\right\}$$

- Final probability:

$$\Pr = \frac{\binom{10}{6}2^6}{\binom{20}{6}} = 0.34$$

Combinatorial Methods
“The neighbour problem”

- Eight important heads of state, the US president and the British prime minister are present at a summit. At the photo the dignataries are lined up randomly.
- Compute the probability that they are next to each other.